Geometry, Analysis and Computation for Network Systems

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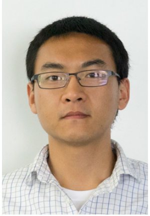
Acknowledgments

Saber Jafarpour
UCSB

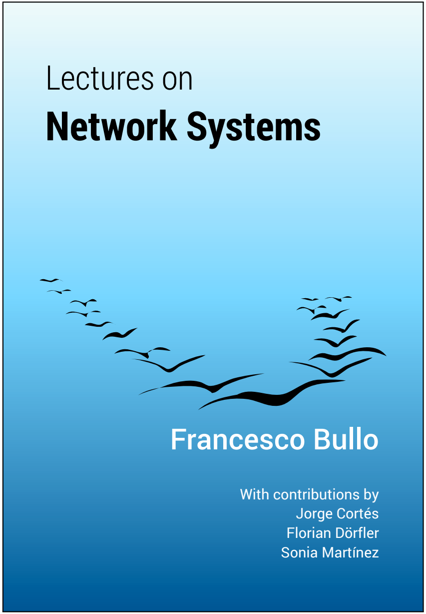


Xiaoming Duan
UCSB

Kevin D. Smith
UCSB

NSF AFOSR ARO ONR DOE

Lectures on Network Systems



Lectures on Network Systems, Francesco Bullo, Createspace, 1 edition, 2018, ISBN 978-1-986425-64-3

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Outline

1. **Linear Network Systems and Metzler Matrices**

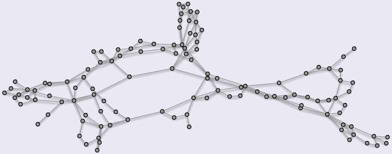
   X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic small gain theorems for Metzler matrices and monotone systems.
   IEEE Transactions on Automatic Control, June 2019.
   Submitted.
   URL: https://arxiv.org/pdf/1905.05868.pdf

2. An emerging theory for Nonlinear Network Systems

3. Kuramoto Synchronization (existence and lack of uniqueness)

Linear network systems

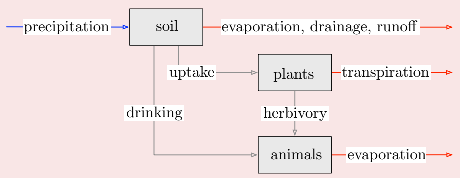
$$\dot{x}(t) = Ax(t)$$



network structure $\Longleftrightarrow$ **function = asymptotic behavior**

Model	Dynamics	Asy Behavior	Graph property
averaging flow (Abelson '64)	$\dot{x} = -Lx$ Laplacian matrix	consensus	$\exists$ globally reach node
network flow (Noy Meir '73)	$\dot{x} = -L^\top x$ transpose Laplacian	stationary distribution	$\exists$ globally reach node
network flow with decay (outflows)	$\dot{x} = Cx$ $C = -L^\top - \text{diag}(d)$ compartmental matrix	stability	outflow-connected
network flow with decay/growth	$\dot{x} = Mx$ $M = -L^\top + \text{diag}(g - d)$ Metzler matrix	stability	**unknown**

Network flow systems



Water flow model for a desert ecosystem (Noy-Meir '73)

$$\dot{q}_i = \sum_j (F_{j\to i} - F_{i\to j}) - F_{i\to 0} + u_i$$

$F_{i\to j} = f_{ij} q_i, \quad F = [f_{ij}]$

$$\dot{q} = \underbrace{\left(F^T - \mathrm{diag}(F\mathbb{1}_n + f_0)\right)}_{=:\, C} q + u$$

C **compartmental matrix**:

quasi-positive (off-diag ≥ 0) and non-positive column sums ($f_0 \geq 0$)

analysis tools: *PF for quasi-positive, inverse positivity, algebraic graph*

system (= each condensed sink) is outflow-connected $\Longleftrightarrow$ C is Hurwitz

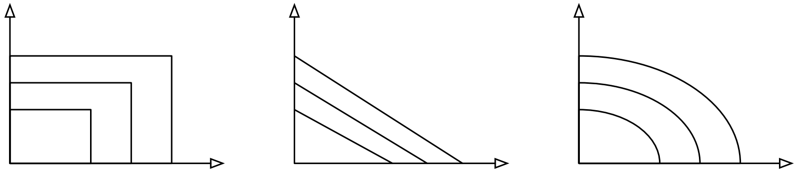
$\Rightarrow$ $\lim_{t\to\infty} q(t) = -C^{-1}u \geq 0$

$(-C^{-1}u)_i > 0 \iff i$th compartment is inflow-connected

Stability of network flow systems

A Metzler M is Hurwitz iff any following equivalent condition hold:

1. there exists $\xi \in \mathbb{R}^n$ such that $\xi > \mathbb{0}_n$ and $M\xi < \mathbb{0}_n$;
2. there exists $\eta \in \mathbb{R}^n$ such that $\eta > \mathbb{0}_n$ and $\eta^\top M < \mathbb{0}_n^\top$; or
3. there exists a diagonal matrix $P \succ 0$ such that $M^\top P + PM \prec 0$.

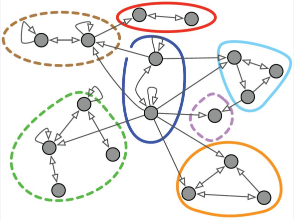


(a) $\max_{i \in \{1,\dots,n\}} x_i/\xi_i$ (b) $\eta^\top x$ (c) $x^\top P x$

Goal: graph-theoretic conditions for stability

Reducible and acyclic graphs

Reducible graphs



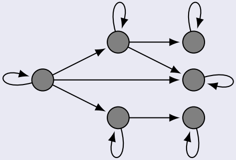
$M \in \mathbb{R}^{n \times n}$ is Hurwitz

$\Updownarrow$

Strongly connected components are Hurwitz

Implication: large-scale system may be decomposed into smaller systems

Directed acyclic graphs



$M \in \mathbb{R}^{n \times n}$ is Hurwitz

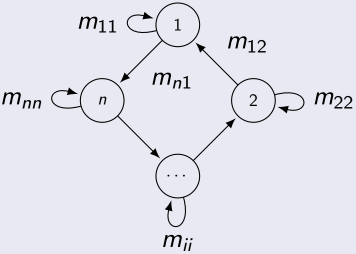
$\Updownarrow$

diagonal entries are negative

Implication: study cycles!

Basic ideas: a simple cycle

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & \cdots & 0 \\ 0 & m_{22} & m_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n1} & 0 & \cdots & 0 & m_{nn} \end{bmatrix}$$



$$M \text{ Hurwitz} \iff \left(\frac{m_{12}}{-m_{11}}\right)\left(\frac{m_{23}}{-m_{22}}\right)\cdots\left(\frac{m_{n1}}{-m_{nn}}\right) < 1$$

where

- $\frac{m_{ij}}{-m_{ii}}$ represents a "gain" for subsystem i with respect to j
- test: composition of "gains" along the cycle is less than 1

Basic ideas: Small-gain network stability

Cyclic Small-Gain Theorem

a network of systems with input is ISS if

$$\text{cycle gain } < 1$$

about each simple cycle,
for appropriate interconnection gains

1. V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems.* Kluwer Academic Publishers, 1991
2. S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010. doi:10.1137/090746483
3. T. Liu, D. J. Hill, and Z.-P. Jiang. Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks. *Automatica*, 47(9):2088–2093, 2011.

Summary of results

Thm 1: Input-to-state interconnection gains for Metzler systems

Thm 2: Max-interconnection gains and graph-theoretic conditions

Thm 3: Sum-interconnection gains and graph-theoretic conditions

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic small gain theorems for Metzler matrices and monotone systems.
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Possible notions of ISS gains

An interconnected nonlinear system with subsystem dynamics

$$\dot{x}_i = f_i(x_i, x_{\mathcal{N}_i}, u_i), \qquad \forall i \in \{1, \ldots, n\}.$$

system has sum-interconnection gains $\{\gamma_{ij}\}$ if

$$|x_i(t)| \;\le\; \beta_i(|x_i(0)|, t) \;+\; \sum_{j \in \mathcal{N}_i} \gamma_{ij}(\|x_j\|_{[0,t]}) \;+\; \gamma_i(\|u_i\|_\infty).$$

where $\beta_i \in \mathcal{KL}$, $\gamma_{ij} \in \mathcal{K}$, and $\gamma_i \in \mathcal{K}$.

system has max-interconnection gains $\{\psi_{ij}\}$ if

$$|x_i(t)| \;\le\; \max_{j \in \mathcal{N}_i} \left\{ \beta_i'(|x_i(0)|, t),\; \psi_{ij}(\|x_j\|_{[0,t]}),\; \psi_i(\|u_i\|_\infty) \right\}.$$

where $\beta_i \in \mathcal{KL}$, $\psi_{ij} \in \mathcal{K}$, and $\psi_i \in \mathcal{K}$.

Thm 1: ISS gains for Metzler systems

Thm 1: ISS gains for Metzler systems

For Metzler system $\dot{x} = Mx + u$, M with negative diagonals,

1. sum-interconnection gains $\{\gamma_{ij}\}$ satisfy

$$\frac{m_{ij}}{-m_{ii}} \leq \gamma_{ij}, \quad \forall i \in \{1, \ldots, n\}, j \in \mathcal{N}_i$$

2. max-interconnection gains $\{\psi_{ij}\}$ satisfy

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \quad \forall i \in \{1, \ldots, n\}$$

For $c = (i_1, i_2, \ldots, i_k, i_1)$ be a simple cycle

1. the sum-cycle gain of c is $\gamma_c = (\gamma_{i_2 i_1})(\gamma_{i_3 i_2}) \ldots (\gamma_{i_1 i_k})$
2. a max-cycle gain of c is $\psi_c = (\psi_{i_2 i_1})(\psi_{i_3 i_2}) \ldots (\psi_{i_1 i_k})$

Thm 2: Max-cycle gains and graph conditions

Thm 2: Conditions based on max-cycle gains

Given an irreducible Metzler matrix $M \in \mathbb{R}^{n \times n}$ with negative diagonal elements and the set of simple cycles Φ, the followings are equivalent:

1. M is Hurwitz;
2. for every $i \in V$ and $j \in \mathcal{N}_i$, there exists $\psi_{ij} > 0$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}} \right) \psi_{ij}^{-1} < 1, \qquad \forall i \in \{1, \dots, n\},$$
$$\psi_c < 1, \qquad \forall c \in \Phi.$$

- "cycle gain < 1 about each simple cycle" is now IFF
- convex problem

Thm 3: Sum-cycle gains and graph conditions

Thm 3: Conditions based on sum-cycle gains

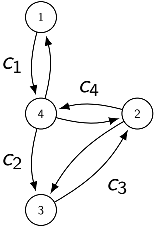
Given an irreducible Metzler matrix $M \in \mathbb{R}^{n \times n}$ with negative diagonal elements, the followings are equivalent:

1. M is Hurwitz;
2. for each i, let Φ_i be simple cycles over $\{1, \ldots, i\}$ (or renumbered)

$$\sum_{c_1 \in \Phi_i} \gamma_{c_1} - \sum_{\substack{\{c_1, c_2\} \subset \Phi_i \\ c_1 \cap c_2 = \emptyset}} \gamma_{c_1} \gamma_{c_2} + \cdots + \sum_{\substack{\{c_1, \ldots, c_{r_i}\} \subset \Phi_i \\ c_i \cap c_j = \emptyset}} (-1)^{r_i - 1} \gamma_{c_1} \ldots \gamma_{c_{r_i}} < 1$$

- condition 2 $\iff$ certain sums of products of gains < 1
- computation of sum-cycle gains and "sums of products" is straightforward (not iterative)

Thm 3: Example



$$
\begin{aligned}
V_1 = \{1\} &\implies \emptyset \\
V_2 = \{1,4\} &\implies \{\gamma_{c_1} < 1\} \\
V_3 = \{1,4,2\} &\implies \{\gamma_{c_1} + \gamma_{c_4} < 1\} \\
V_4 = \{1,4,2,3\} &\implies \{\gamma_{c_1} + \gamma_{c_4} < 1, \\
&\qquad \gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} + \gamma_{c_4} - \gamma_{c_1}\gamma_{c_3} < 1\}
\end{aligned}
$$

Hence, stability certificate

$$
\begin{aligned}
\gamma_{c_1} + \gamma_{c_4} &< 1 \\
\gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} + \gamma_{c_4} - \gamma_{c_1}\gamma_{c_3} &< 1
\end{aligned}
$$

Outline

1. Linear Network Systems and Metzler Matrices

2. An emerging theory for Nonlinear Network Systems

   F. Bullo. *Lectures on Network Systems*.
   Kindle Direct Publishing, 1.3 edition, July 2019.
   With contributions by J. Cortés, F. Dörfler, and S. Martínez.
   URL: http://motion.me.ucsb.edu/book-lns

3. Kuramoto Synchronization (existence and lack of uniqueness)

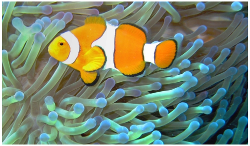
Nonlinear network systems

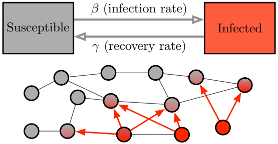
Rich variety of emerging behaviors

1. equilibria / limit cycles / extinction in populations dynamics
2. epidemic outbreaks in spreading processes
3. synchrony and multi-stability in coupled oscillators

Rich variety of analysis tools

1. nonlinear stability theory
2. passivity, small gain theorems, and dissipativity
3. contractivity and monotonicity





Example: Population systems in ecology
(Vito Volterra, Universita' di Torino, 1860-1940)

Mutualism clownfish / anemones (Takeuchi et al '78)

Lotka-Volterra: x_i = quantity/density

$$\frac{\dot{x}_i}{x_i} = b_i + \sum_j a_{ij} x_j$$

$$\dot{x} = \mathrm{diag}(x)\big(Ax + b\big)$$

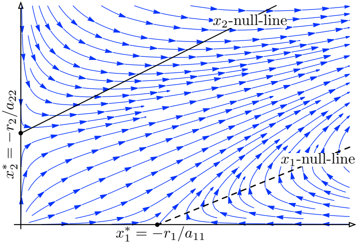
interaction matrix A:

$(+,+)$ mutualism, $(+,-)$ predation, $(-,-)$ competition
rich behavior: persistence, extinction, equilibria, periodic orbits, . . .

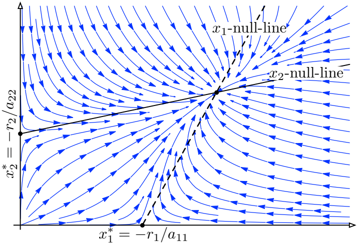
1. **mutualism:** $a_{ij} \geq 0$
2. either unbounded evolution or

   exists unique steady state $-A^{-1}b > 0$
   $\lim_{t\to\infty} x(t) = -A^{-1}b$ from all $x(0) > 0$

Dichotomy in mutualistic Lotka-Volterra system



Case I: $a_{12} > 0$, $a_{21} > 0$, $a_{12}a_{21} > a_{11}a_{22}$. There exists no positive equilibrium point. All trajectories starting in $\mathbb{R}^2_{>0}$ diverge.



Case II: $a_{12} > 0$, $a_{21} > 0$, $a_{12}a_{21} < a_{11}a_{22}$. There exists a unique positive equilibrium point. All trajectories starting in $\mathbb{R}^2_{>0}$ converge to the equilibrium point.

Research questions in Nonlinear Network Systems

1. what are key example systems?
2. what is a useful underlying structure?

3. what is a practical, simple, rich technical approach?
4. how do we treat dichotomy and richer behaviors?
5. how do we automatically generate Lyapunov functions?

Example systems

Kuramoto oscillators ('75)

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Metzler Jac: phase cohesive region

Ex: active power flow, motion patterns

Yorke network propagation ('76)

$$\dot{x} = \beta\big(I_n - \operatorname{diag}(x)\big)Ax - \gamma x$$

Metzler Jac and positive

Ex: network SIR, patchy SIS

Lotka-Volterra population ('20)

$$\dot{x} = \operatorname{diag}(x)\big(Ax + r\big)$$

Metzler Jac: mutualistic interactions

Ex: biochemical networks, repressilator with 2 genes

Daganzo cell transmission ('94)

$$\dot{\rho}_e = f_e^{\text{in}}(\rho) - f_e^{\text{out}}(\rho)$$

Metzler Jac: free flow region

Ex: monotone distributed routing (Como, Savla, et al), Maeda '78, Sandberg '78

Matrosov interconnection of ISS systems ('71)

$$\dot{x}_i = f_i(x_1, \dots, x_n, u_i) \quad \Longrightarrow \quad \dot{v} \leq -A(v) + \Gamma(v) + G(w)$$

Metzler Jac and positive

A review of Contraction Theory

given norm, the **matrix measure** of A is

$$\mu(A) := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

assume: vector field f is **infinitesimally contracting** over C, that is,

$$\mu\big(Df(x)\big) \leq c < 0, \quad \text{for all } x \in C$$

assume: set C is **f-invariant**, closed and convex

Desirable consequences

1. flow of f is a contraction, i.e.,
   distance between solutions exponentially decreases with rate c
2. there exists an equilibrium x^*, unique, globally exponentially stable with global Lyapunov functions

$$x \mapsto \|x - x^*\|^2 \quad \text{and} \quad x \mapsto \|f(x)\|^2$$

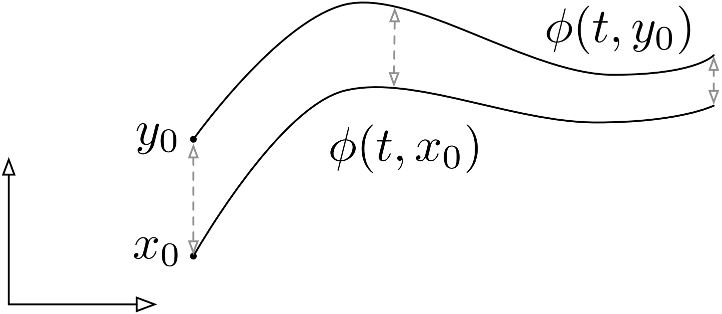


Figure: Any two trajectories of an infinitesimally contracting system converge.

Common matrix measures

Vector norm	Matrix measure
$\lVert x \rVert_1 = \sum_{i=1}^{n} \lvert x_i \rvert$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1, i\neq j}^{n} \lvert a_{ij} \rvert \right)$ $= \max \text{ column ``absolute sum'' of } A$
$\lVert x \rVert_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$	$\mu_2(A) = \lambda_{\max}\left(\frac{A + A^\top}{2}\right)$
$\lVert x \rVert_\infty = \max_{i\in\{1,\dots,n\}} \lvert x_i \rvert$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1, j\neq i}^{n} \lvert a_{ij} \rvert \right)$ $= \max \text{ row ``absolute sum'' of } A$

Simplifications for a Metzler matrix M

$$\mu_1(M) = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^{n} m_{ij} = \max(M^\top \mathbb{1}_n) = \max \text{ column sum of } M$$

$$\mu_\infty(M) = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} m_{ij} = \max(M \mathbb{1}_n) = \max \text{ row sum of } M$$

The Euclidean case: works by Krasovskiĭ & Vidyasagar

Vidyasagar '78: Lyapunov functions and matrix measures

Given $P \succ 0$ and $c \in \mathbb{R}$,

$$\mu_{2,P}(A) < c \qquad \iff \qquad A^\top P + PA \prec 2cP$$

1. A Hurwitz $\iff$ A has negative weighted 2-norm (w.r.t. some P)
2. $\inf_{P \succ 0} \mu_{2,P}(A) =$ spectral abscissa of A

Krasovskiĭ '60: method to design Lyapunov function

f is weighted 2-norm contracting if $\exists P \succ 0$ and $c < 0$

$$P\,Df(x) + Df(x)^\top P \preceq 2cP, \qquad \text{for all } x \in \mathbb{R}^n$$

Constant Lyapunov weight P at each x implies <u>desirable consequences</u>

The non-Euclidean case for Metzler Jacobians

Coogan '16: matrix measures of a Metzler matrix M

Given vectors $\eta, \xi > \mathbb{0}_m$ and $c \in \mathbb{R}$,

$$\mu_{1,\mathrm{diag}(\eta)}(M) < c \iff \eta^\top M < c\eta^\top, \text{ and}$$
$$\mu_{\infty,\mathrm{diag}(\xi)^{-1}}(M) < c \iff M\xi < c\xi,$$

1. M Hurwitz $\iff$ M has negative weighted 1- or ∞-measure
2. $\inf_{\eta > \mathbb{0}_m} \mu_{1,\mathrm{diag}(\eta)}(M) = \inf_{\xi \mathbb{0}_m} \mu_{\infty,\mathrm{diag}(\xi)^{-1}}(M) =$ spectral abscissa of M

Sum-separable and max-separable Lyapunov functions

f with Metzler Jac is weighted 1-norm contracting if $\exists \eta > \mathbb{0}_n$ and $c < 0$

$$\eta^\top Df(x) \leq c\eta^\top, \quad \text{for all } x \in \mathbb{R}^n$$

Constant column weights η at each x implies desirable consequences

Krasovskiĭ Lyapunov functions for systems with Metzler Jacobians and constant weights

Weighted diagonal 2-norm:

$$\|x - x^*\|_P^2 = \sum_{i=1}^{n} p_i (x_i - x_i)^2 \quad \text{and} \quad \|f(x)\|_P^2 = \sum_{i=1}^{n} p_i f_i(x)^2$$

Weighted 1-norm

$$\|x - x^*\|_{1,\eta} = \sum_{i=1}^{n} \eta_i |x_i - x_i^*| \quad \text{and} \quad \|f(x)\|_{1,\eta} = \sum_{i=1}^{n} \eta_i |f_i(x)|$$

Weighted ∞-norm

$$\|x - x^*\|_{\infty,\xi^{-1}} = \max_{i \in \{1,\dots,n\}} \frac{|x_i - x_i^*|}{\xi_i} \quad \text{and} \quad \|f(x)\|_{\infty,\xi^{-1}} = \max_{i \in \{1,\dots,n\}} \frac{|f_i(x)|}{\xi_i}$$

Recall: sublevel sets of Lyapunov functions are f-invariant

Example application to Lotka-Volterra

1. change of variable $y = \ln x$, so that $x \in \mathbb{R}^n_{>0}$ maps into $y \in \mathbb{R}^n$ and

$$\dot{y} = A\exp(y) + r := f_{\mathrm{LVe}}(y)$$

2. pick $v > \mathbb{0}_n$ such that $v^\top A < \mathbb{0}_n$ and show

$$v^\top Df_{\mathrm{LVe}}(y) = v^\top A\,\mathrm{diag}(\exp(y)) < -cv^\top\,\mathrm{diag}(\exp(y)) \leq 0.$$

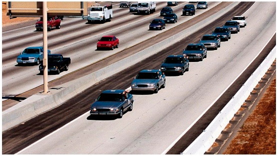
3. f_{LVe}, and so f_{LV}, has a unique globally exponentially stable equilibrium with sum-separable global Lyapunov functions

$$\|y - y^*\|_{1,\mathrm{diag}(v)} \quad \text{and} \quad \|f_{\mathrm{LVe}}(y)\|_{1,\mathrm{diag}(v)}$$

that is,

$$x \mapsto \sum_{i=1}^{n} v_i|\ln(x_i/x_i^*)|, \qquad x \mapsto \sum_{i=1}^{n} v_i|(Ax+r)_i|$$

Why is this relevant for infrastructure networks?

Consider a network flow system $\dot{x} = f(x)$ preserving a commodity

$$\begin{aligned} \text{constant} &= \mathbb{1}_n^\top x(t) \\ &\implies 0 = \mathbb{1}_n^\top \dot{x}(t) = \mathbb{1}_n^\top f(x(t)) \\ &\implies \mathbb{0}_n = \mathbb{1}_n^\top Df\, x(t) \end{aligned}$$

If additionally f has Meztler Jacobian, then f is automatically weakly contracting (non-expansive) with respect to the ℓ_1 norm.

Outline

1. Linear Network Systems and Metzler Matrices

2. An emerging theory for Nonlinear Network Systems

3. Kuramoto Synchronization (existence)

   S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections.
   IEEE Transactions on Automatic Control, 64(7):2830–2844, 2019.
   doi:10.1109/TAC.2018.2876786

   1. problem statement
   2. solution

4. Kuramoto Multi-Stability (lack of uniqueness)

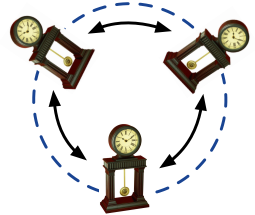
   S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. Multistable synchronous power flows: From geometry to analysis and computation.
   SIAM Review, January 2019.
   Submitted.
   URL: https://arxiv.org/pdf/1901.11189.pdf

Today: Sync & Multi-Stability in Coupled Oscillators

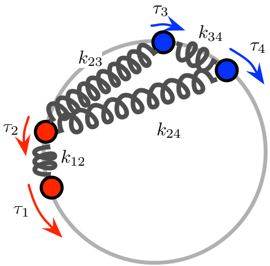
Kuramoto model

- n **oscillators** with angle $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- **coupling** with strength $a_{ij} = a_{ji}$

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



Model #1: Spring network analog and applications



Coupled swing equations

Euler-Lagrange eq for spring network on ring:

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \tau_i - \sum_j k_{ij}\sin(\theta_i - \theta_j)$$

Kuramoto coupled oscillators

$$\dot{\theta}_i = \omega_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$$

Kuramoto equilibrium equation

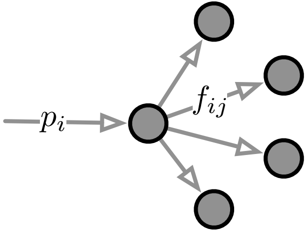
$$0 = \omega_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$$

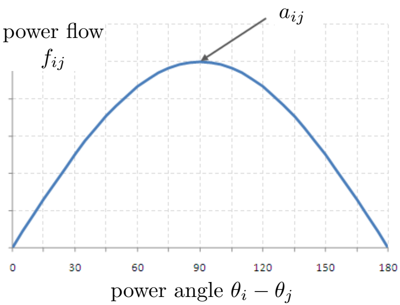
Model #2: Active Power Flow Problem

AC, Kirchhoff and Ohm, quasi-sync, lossless lines, constant voltages.

supply/demand p_i, max power coeff a_{ij}, voltage phase θ_i

$$p_i = \sum_{j=1}^{n} f_{ij}, \qquad f_{ij} = a_{ij} \sin(\theta_i - \theta_j)$$



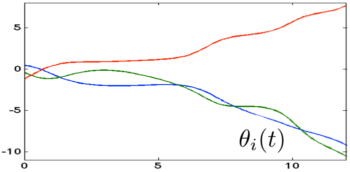


Given: network parameters & topology, load & generation profile,

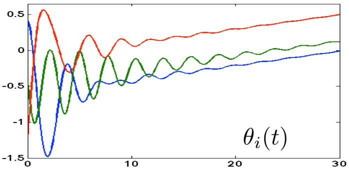
Phenomenon #1: Transition from incoherence to sync

Function = synchronization

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



large $|\omega_i - \omega_j|$ & small coupling
$\Rightarrow$ incoherence = no sync



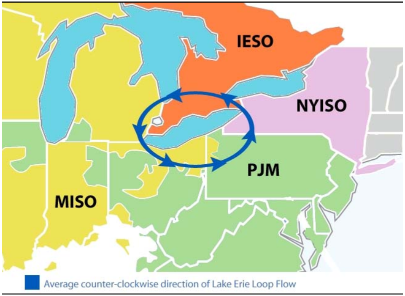
small $|\omega_i - \omega_j|$ & large coupling
$\Rightarrow$ coherence = frequency sync

Phenomenon #2: Multiple power flows

Theoretical observation: multiple solutions exist

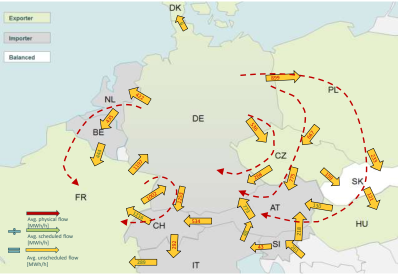
Practical observations:

- sometimes undesirable power flows around loops
- sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, **Lake Erie Loop Flow Mitigation**, Technical Report, 2008

Figure 8: Average unscheduled flows for the years 2011 and 2012, MWh/h[8]



Source: THEMA Consulting Group, based on data from 16 TSOs

THEMA Consulting Group, **Loop-flows - Final advice**, Technical Report prepared for the European Commission, 2013

Outline

1. Linear Network Systems and Metzler Matrices

2. An emerging theory for Nonlinear Network Systems

3. Kuramoto Synchronization (existence)

   S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections. *IEEE Transactions on Automatic Control*, 64(7):2830–2844, 2019. doi:10.1109/TAC.2018.2876786

   1. problem statement
   2. solution

4. Kuramoto Multi-Stability (lack of uniqueness)

   S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. Multistable synchronous power flows: From geometry to analysis and computation. *SIAM Review*, January 2019. Submitted. URL: https://arxiv.org/pdf/1901.11189.pdf

Primer on algebraic graph theory (slide 1/2)

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p_{\text{actv}})_{(ij)} = p_i - p_j$

Weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian stiffness: $L = B\mathcal{A}B^\top \geq 0$

Linearization of Kuramoto equilibrium equation:

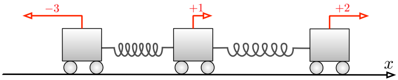
$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta) \quad \Longrightarrow \quad p_{\text{actv}} \approx B\mathcal{A}(B^\top\theta) = L\,\theta$$

Algebraic connectivity:

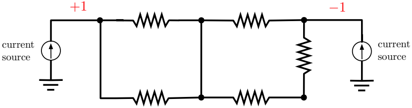
$$\begin{aligned}\lambda_2(L) &= \text{second smallest eig of } L \\ &= \text{notion of connectivity and coupling}\end{aligned}$$

Primer on algebraic graph theory (slide 2/2)

Laplacian linear balance equation



(a) spring network



(b) resistive circuit

$$L_{\text{stiffness}}\, x = f_{\text{load}} \qquad \text{and} \qquad L_{\text{conductance}}\, v = c_{\text{injected}}$$

Laplacian linear balance equation: $p_{\text{actv}} = L\,\theta$

if $\sum_i p_i = 0$ in $p_{\text{actv}} = L\,\theta$, then equilibrium exists : $\theta = L^{\dagger} p_{\text{actv}}$

pairwise displacements : $B^{\top}\theta = B^{\top} L^{\dagger} p_{\text{actv}}$

From Old to New Tests

Question: Given balanced p_{actv}, do angles exist satisfying

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top \theta)$$

Old Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top p_{\text{actv}}\|_2 < \lambda_2(L) \quad \text{for unweighted graphs} \quad \text{(Old 2-norm T)}$$

$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < 1 \quad \text{for trees, complete} \quad \text{(Old } \infty\text{-norm T)}$$

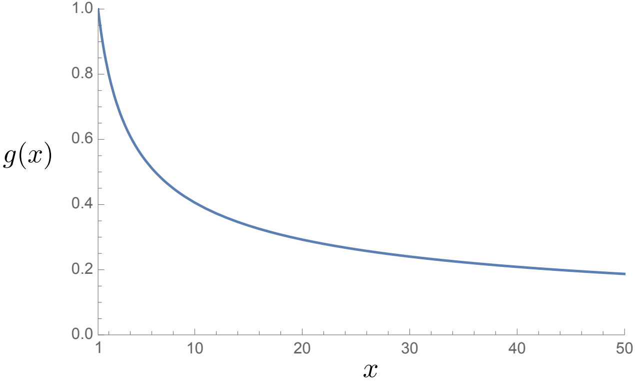
New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top L^\dagger p_{\text{actv}}\|_2 < 1 \quad \text{for unweighted graphs} \quad \text{(New 2-norm T)}$$

$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < g(\|\mathcal{P}\|_\infty) \quad \text{for all graphs} \quad \text{(New } \infty\text{-norm T)}$$

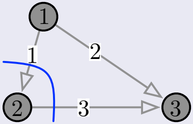
where g is monotonically decreasing

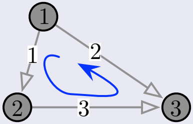
$$g : [1, \infty) \to [0, 1]$$
$$x \mapsto \frac{y(x) + \sin(y(x))}{2} - x\,\frac{y(x) - \sin(y(x))}{2}\Big|_{y(x) = \arccos(\frac{x-1}{x+1})}$$



and where $\mathcal{P}$ is a projection matrix

$\mathcal{P} = B^\top L^\dagger B\mathcal{A}$ $\quad = $ oblique projection onto $\mathrm{Im}(B^\top)$ parallel to $\mathrm{Ker}(B\mathcal{A})$





$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\mathrm{Im}(B^\top)}_{\substack{\text{cutset space}\\ \text{flow vectors}}} \oplus \underbrace{\mathrm{Ker}(B\mathcal{A})}_{\substack{\text{weighted cycle space}\\ \text{cycle vectors}}}$$

1. if G unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$

New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top L^\dagger p_{\mathrm{actv}}\|_2 < 1$	for unweighted graphs	(New 2-norm T)
$\|B^\top L^\dagger p_{\mathrm{actv}}\|_\infty < g(\|\mathcal{P}\|_\infty)$	for all graphs	(New ∞-norm T)

Unifying theorem with a family of tests

Equilibrium angles (neighbors within γ arc) exist if, in some p-norm,

$$\|B^\top L^\dagger p_{\mathrm{actv}}\|_p \leq \gamma \alpha_p(\gamma) \qquad \text{for all graphs} \qquad (\text{New } \alpha_p \text{ T})$$

where nonconvex optimization problem:

$$\alpha_p(\gamma) := \text{ min amplification factor of } \quad \mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)]$$

Proof sketch 1/2: Rewriting the equilibrium equation

For what $B, \mathcal{A}, p_{\text{actv}}$ does there exist θ solution to:

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta)$$

STEP 1: For what flow z and projection $\mathcal{P}$ onto cutset/flow space, does there exist a flow x that solves

$$\mathcal{P}\sin(x) = z$$

$$\iff \quad \mathcal{P}\,\text{diag}[\text{sinc}(x)]x = z$$

$$\iff \quad x = (\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}z =: h(x)$$

Proof sketch 2/2: Amplification factor & Brouwer

STEP 1: look for x solving

$$x = h(x) = (\mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)])^{-1} z$$

IDEA: assume $\|x\|_p \leq \gamma$ and ensure $\|h(x)\|_p \leq \gamma$

STEP 2: if one defines min amplification factor

$$\alpha_p(\gamma) := \min_{\|x\|_p \leq \gamma} \min_{\|y\|_p = 1} \|\mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)] y\|_p$$

then $$\|h(x)\|_p \leq \max_x \max_y \|(\mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)])^{-1} y\|_p \cdot \|z\|_p$$

$$= \left(\min_x \min_y \|\mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)] y\|_p\right)^{-1} \|z\|_p \quad \leq \frac{\|z\|_p}{\alpha_p(\gamma)}$$

STEP 3: $\|z\|_p \leq \gamma \alpha_p(\gamma)$, then $\|h(x)\|_p \leq \gamma$ so that h satisfies Brouwer

Comparison of sufficient and approximate sync tests

Any test predicts max transmittable power (before bifurcation).
Compare with numerically computed.

Test Case	ratio of test prediction to numerical computation			
	old 2-norm	new ∞-norm	$g(\|\mathcal{P}\|_\infty) \approx 1$ approximate	α_∞ test *fmincon*
IEEE 9	16.54 %	73.74 %	92.13 %	85.06 %†
IEEE 14	8.33 %	59.42 %	83.09 %	81.32 %†
IEEE RTS 24	3.86 %	53.44 %	89.48 %	89.48 %†
IEEE 30	2.70 %	55.70 %	85.54 %	85.54 %†
IEEE 118	0.29 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	29.08 %	82.85 %	—*

† *fmincon* with 100 randomized initial conditions

* *fmincon* does not converge

Summary: Kuramoto equilibrium and active power flow

Given topology (incidence B), admittances (Laplacian L), injections p_{actv},

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Equilibrium angles exist if, in some p-norm,

$$\|B^\top L^\dagger p_{\text{actv}}\|_p \leq \gamma \alpha_p(\gamma) \qquad \text{for all graphs} \qquad (\text{New } \alpha_p \text{ T})$$

For $p = \infty$, after bounding,

$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty \leq g(\|\mathcal{P}\|_\infty) \qquad (\text{New } \infty\text{-norm T})$$

Q1: $\exists$ a **stable operating point** (with pairwise angles $\leq \gamma$)?
Q2: what is the **network capacity** to transmit active power?
Q3: how to quantify **robustness** as distance from loss of feasibility?

Outline

1. ## Introduction to Network Systems

   F. Bullo. *Lectures on Network Systems.*
   Kindle Direct Publishing, 1.3 edition, July 2019.
   With contributions by J. Cortés, F. Dörfler, and S. Martínez.
   URL: http://motion.me.ucsb.edu/book-lns

2. ## Synchronization (existence)

   S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections.
   IEEE Transactions on Automatic Control, 64(7):2830–2844, 2019.
   doi:10.1109/TAC.2018.2876786

3. ## Multi-Stability (lack of uniqueness)

   S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. Multistable synchronous power flows: From geometry to analysis and computation.
   SIAM Review, January 2019.
   Submitted.
   URL: https://arxiv.org/pdf/1901.11189.pdf

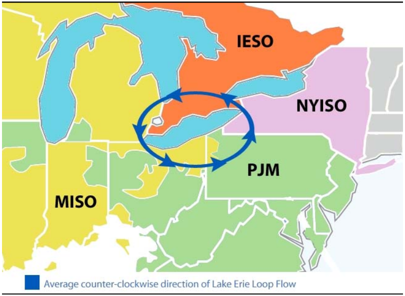
Phenomenon #2: Multiple power flows

Theoretical observation: multiple solutions exist

Practical observations:

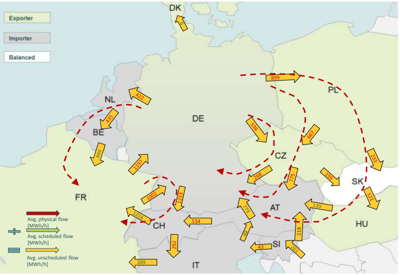
sometimes undesirable power flows around loops

sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, **Lake Erie Loop Flow Mitigation**, Technical Report, 2008

Figure 8: Average unscheduled flows for the years 2011 and 2012, MWh/h[8]



Source: THEMA Consulting Group, based on data from 16 TSOs

THEMA Consulting Group, **Loop-flows - Final advice**, Technical Report prepared for the European Commission, 2013

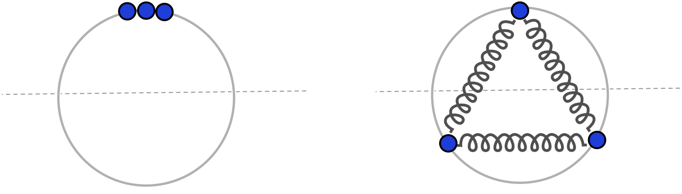
Lack of uniqueness and winding solutions

Given topology (incidence B), admittances (Laplacian L), injections p_{actv},

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. is solution unique?
2. how to localize/classify solutions?

triangle graph, homogeneous weights ($a_{ij} = 1$), $p_{\text{actv}} = 0$



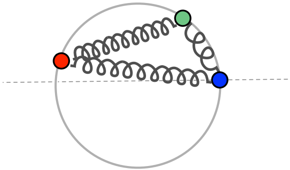
phase sync　　　　splay state

Winding number of n angles

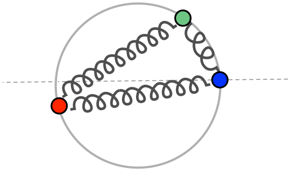
Given undirected graph with a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

1. **winding number of $\theta \in \mathbb{T}^n$ along σ** is:

$$w_\sigma(\theta) = \frac{1}{2\pi} \sum_{i=1}^{n_\sigma} d_{cc}(\theta_i, \theta_{i+1})$$



$w(\theta) = 0$



$w(\theta) = \pm 1$

2. given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of θ** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

"Kirchhoff Angle Law" and partition of the n-torus

Theorem: Kirchhoff angle law on $\mathbb{T}^n$

$$w_\sigma(\theta) = 0, \pm 1, \dots, \pm \lfloor n_\sigma/2 \rfloor$$
$$\implies w(\theta) \text{ is piecewise constant}$$
$$\implies w(\theta) \text{ takes value in a finite set}$$

Theorem: Winding partition

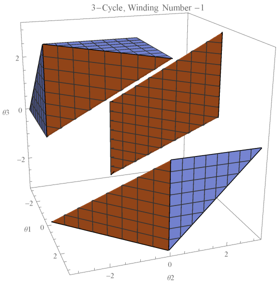
For each possible winding vector u, define

$$\text{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

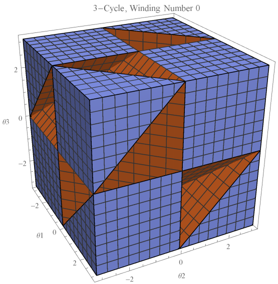
Then

$$\mathbb{T}^n = \cup_u \text{WindingCell}(u)$$

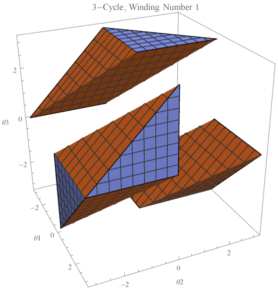
Winding partition of triangle graph



$w = -1$

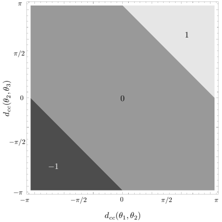


$w = 0$



$w = +1$

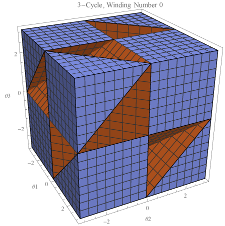
- each winding cell is connected
- each winding cell is invariant under rotation
- bijection:
  reduced winding cell $\longleftrightarrow$ open convex polytope



The Kuramoto model and the winding partition

Given topology (incidence B), admittances (Laplacian L), injections p_{actv},

$$\dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$



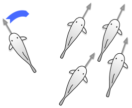
Theorem: At-most-uniqueness and extensions

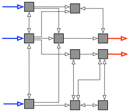
1. each WindingCell has at-most-unique equilibrium with $\Delta\theta < \pi/2$
2. equilibrium loop flow increases monotonically wrt winding number
3. existence + uniqueness in WindingCell(u) with $\Delta\theta < \pi/2$ if

$$\|B^\top L^\dagger p_{\text{actv}} + Cu\|_\infty \leq g(\|\mathcal{P}\|_\infty), \text{ or} \qquad \text{(Static T)}$$

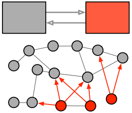
$\exists$ a trajectory inside WindingCell(u) with $\Delta\theta < \pi/2$ (Dynamic T)

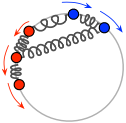
Summary and Future Work

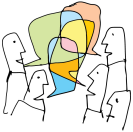

averaging


compartmental flows


mutualism


virus spread


coupled oscillators


social systems

Review

1. a rather comprehensive theory of linear network systems
2. an emergent theory of nonlinear network systems based on contractivity and monotonicity
3. existence and multistability for Kuramoto

Future research

1. a little bit more on Metzler matrices
2. much work on monotonicity and contractivity
3. applications to other dynamic flow networks
4. **outreach/collaboration opportunities for our community with**
   sociologists, biologists, economists, physicists ...