The University of Texas at Austin
Civil, Architectural and
Environmental Engineering
Cockrell School of Engineering

Robust control of traffic flow over networks using chance-constrained optimization

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Outline

Motivation

Traffic flow control of highways using Hamilton-Jacobi equations

- Background
- Problem definition
- Optimization formulations

Robust flow control problems on single links using chance constraints

- Initial condition uncertainty

Extension to network problems

- Problem formulation
- Simulation results

Conclusion

Motivation

- Highway congestion is a worsening problem in most cities of the world
- Traffic control techniques are relatively inexpensive ways to address traffic congestion (low cost vs. building new roads)
- Various control methods based on PDE flow models (such as the LWR flow model) have been investigated in the past
- Flow control problems are associated with significant uncertainties, including model noise and uncertainty on the initial state of the system

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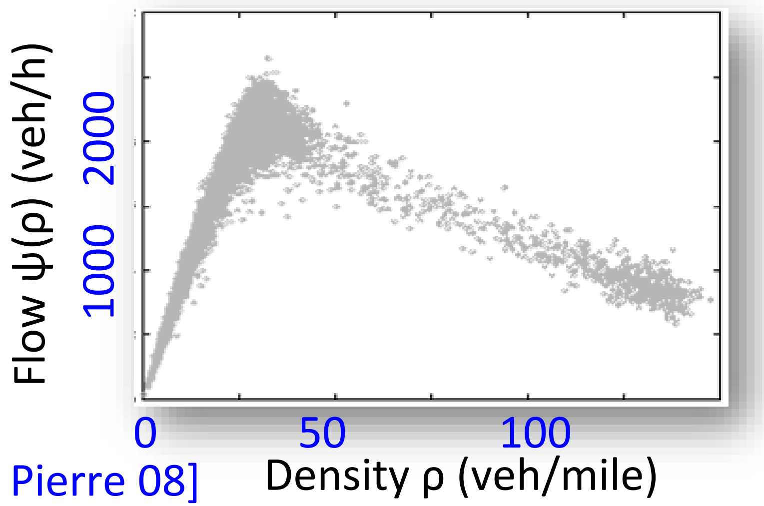
Background

- Traffic flow model (LWR): derived by Lighthill-Whitham (1955), Richards (1956)
- First order scalar hyperbolic conservation law

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial \psi(\rho(t,x))}{\partial x} = 0$$

- Based on two assumptions:
    - conservation of vehicles
    - existence of a relationship between flow and density: $q=\psi(\rho)$

in this problem, ψ(.) is assumed to be concave



[Newell 93], [Daganzo 03,06] [Aubin Bayen Saint Pierre 08]

Hamilton-Jacobi formulation

Equivalently, we can define M(t,x) such that:

$$\mathbf{M}(t_2,x_2)-\mathbf{M}(t_1,x_1)=\int_{x_1}^{x_2}-\rho(t_1,x)dx+\int_{t_1}^{t_2}\psi(\rho(t,x_2))dt$$

The function $M(t,x)$ is the cumulative number of vehicles function, also called *Moskowitz function*. Its spatial derivative is the opposite of the density function; its temporal derivative is the flow function.

Integrating the LWR PDE, $M(t,x)$ solves the Hamilton-Jacobi PDE:

$$\frac{\partial \mathbf{M}(t,x)}{\partial t}-\psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) = 0$$

[Newell 93], [Daganzo 03,06] [Aubin Bayen Saint Pierre 08]

Semi-analytic computational methods

- Many existing computational methods:
- For LWR:
  - Godunov scheme (or equivalently CTM)
  - Wave-front tracking
  - Other finite difference schemes (ENO, WENO)
- For HJ:
  - Lax Friedrichs schemes (or other numerical schemes)
  - Variational method (dynamic programming)
  - Semi-analytic method (for homogeneous problems), which can be used for both HJ and LWR

[Daganzo06] [Claudel Bayen IEEE TAC part I & 2, 2010] [Mazare Dehwah Claudel Bayen, TR-B 2012]

Semi-analytic computational methods

Based on the classical Lax-Hopf formula

For a boundary data function c(.,.), the solution $M_c(.,.)$ is given by:

$$\mathbf{M_c}(t,x) = \inf_{(u,T)\in \mathrm{Dom}(\varphi^*)\times\mathbb{R}_+} \left(\mathbf{c}(t-T, x+Tu) + T\varphi^*(u)\right)$$

where $\varphi^*(u) \;:=\; \sup_{p\in\mathrm{Dom}(\psi)} [p\cdot u + \psi(p)]$ is the convex transform of ψ

Can be solved using dynamic programming on a grid (Variational Theory) [Daganzo06]

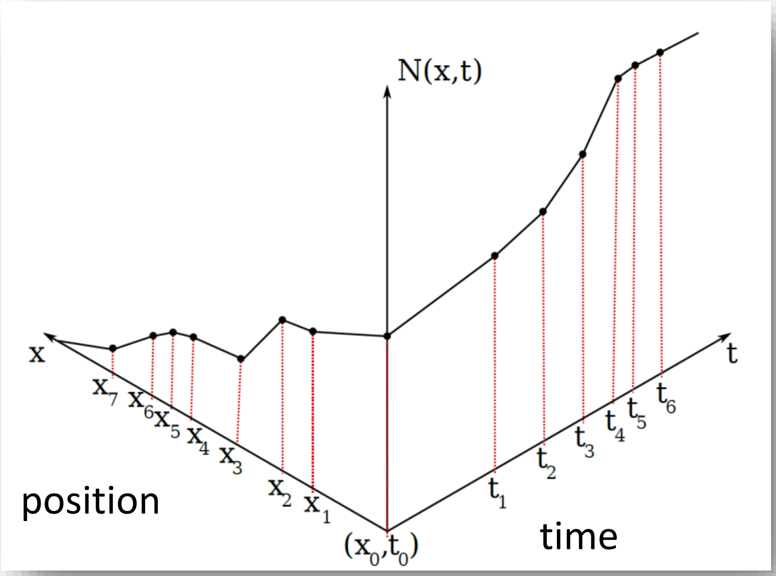
If model parameters are time-space independent, we can exploit the structure of the dynamic programming problem

[Lax 1973] [Aubin Bayen Saint Pierre SIAM SICON 2009] [Claudel Bayen IEEE TAC part II 2010]

Semi-analytic computational methods

We use a piecewise linear decomposition of the boundary data (which amounts to taking piecewise constant initial densities and boundary flows)

Let us compute the solution to a single piece of linear initial condition



[Claudel Bayen IEEE TAC part II 2010] [Mazare Dehwah Claudel Bayen, TR-B 2012]

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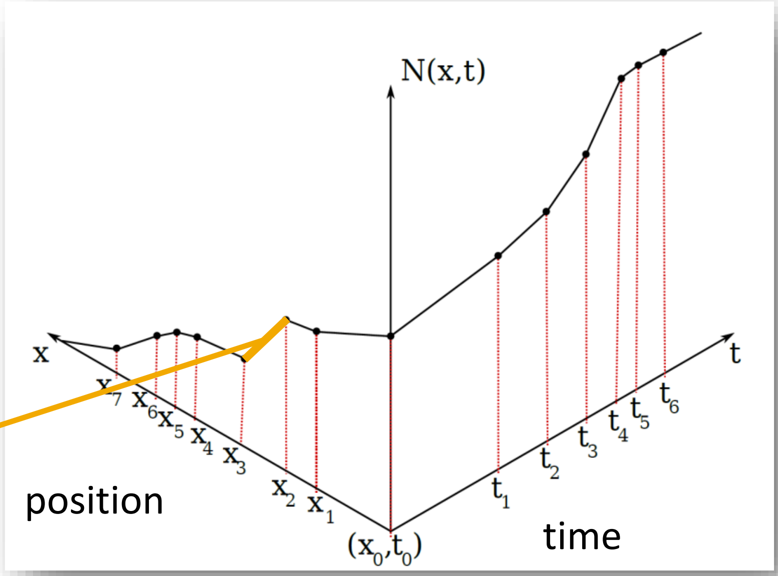
Let us compute the solution to a single piece of linear initial condition

$$\mathbf{M_0}_i(0,x) = \begin{cases} a_i x + b_i & \textit{if } x \in [\overline{\alpha}_i, \overline{\alpha}_{i+1}] \\ +\infty & \textit{otherwise} \end{cases}$$



Physically: constant initial density in a spatial interval
no information elsewhere

[Claudel Bayen IEEE TAC part II 2010] [Mazare Dehwah Claudel Bayen, TR-B 2012]

Semi-analytic computational methods

Single piece of linear initial condition:

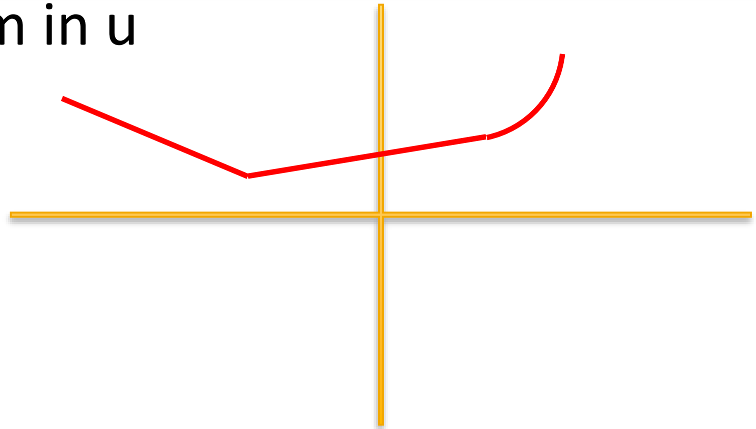
$$\mathbf{M}_{\mathbf{0}_i}(0,x) = \begin{cases} a_i x + b_i & \textit{if } x \in [\overline{\alpha}_i, \overline{\alpha}_{i+1}] \\ +\infty & \textit{otherwise} \end{cases}$$

Associated Lax-Hopf formula:

$$\mathbf{M}_{\mathbf{M}_{\mathbf{0}_i}}(t,x) = \inf_{u \in \mathrm{Dom}(\varphi^*) \cap \left[\frac{\overline{\alpha}_i - x}{t}, \frac{\overline{\alpha}_{i+1} - x}{t}\right]} (a_i(x+tu) + b_i + t\varphi^*(u))$$

1D convex optimization problem in u

Can be solved analytically



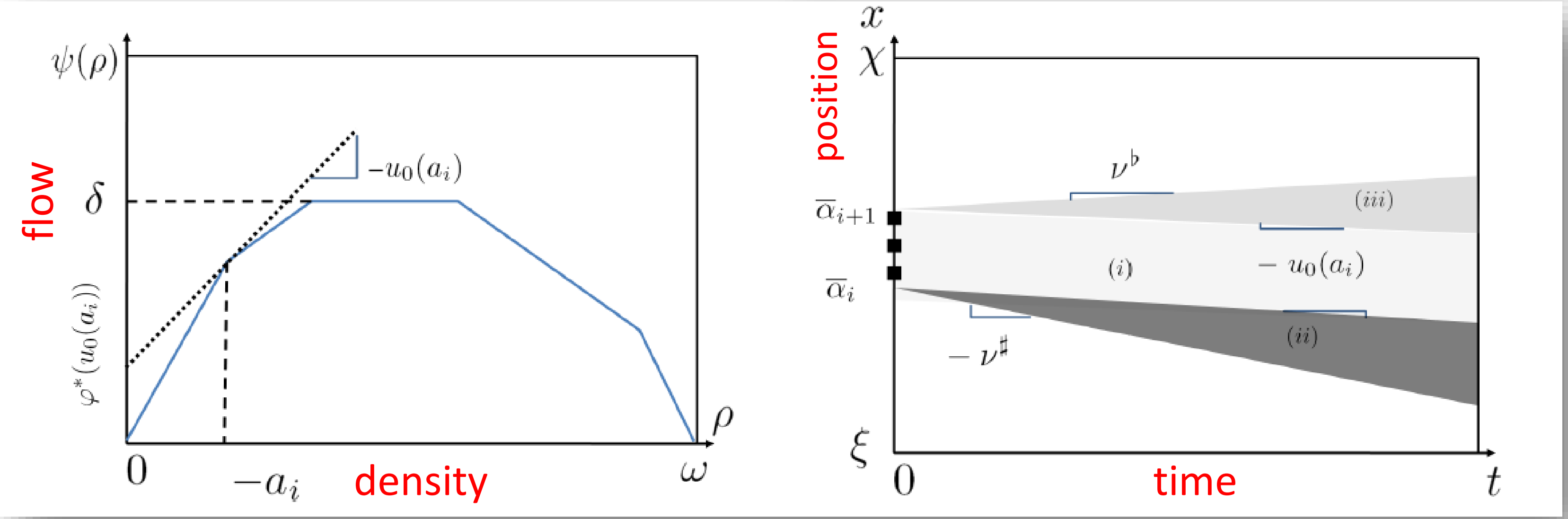
[Claudel Bayen IEEE TAC part II 2010] [Mazare Dehwah Claudel Bayen, TR-B 2012]

Semi-analytic computational methods

<u>Solution structure:</u>

We use subgradients to find the optimum u since ψ is not necessarily differentiable

$$\mathbf{M}_{\mathcal{M}_{0,i}}(t,x) = \begin{cases} (i) & t\psi(-a_i) + a_i x + b_i \\ & \text{if } \ u_0(a_i) \in \left[\frac{\overline{\alpha}_i - x}{t}, \frac{\overline{\alpha}_{i+1} - x}{t}\right] \\ (ii) & a_i \overline{\alpha}_i + b_i + t\varphi^*\left(\frac{\overline{\alpha}_i - x}{t}\right) \\ & \text{if } \ u_0(a_i) \leq \frac{\overline{\alpha}_i - x}{t} \\ (iii) & a_i \overline{\alpha}_{i+1} + b_i + t\varphi^*\left(\frac{\overline{\alpha}_{i+1} - x}{t}\right) \\ & \text{if } \ u_0(a_i) \geq \frac{\overline{\alpha}_{i+1} - x}{t} \end{cases}$$



[Claudel Bayen IEEE TAC part II 2010] [Mazare Dehwah Claudel Bayen, TR-B 2012]

Semi-analytic computational methods

<u>Solution structure:</u>

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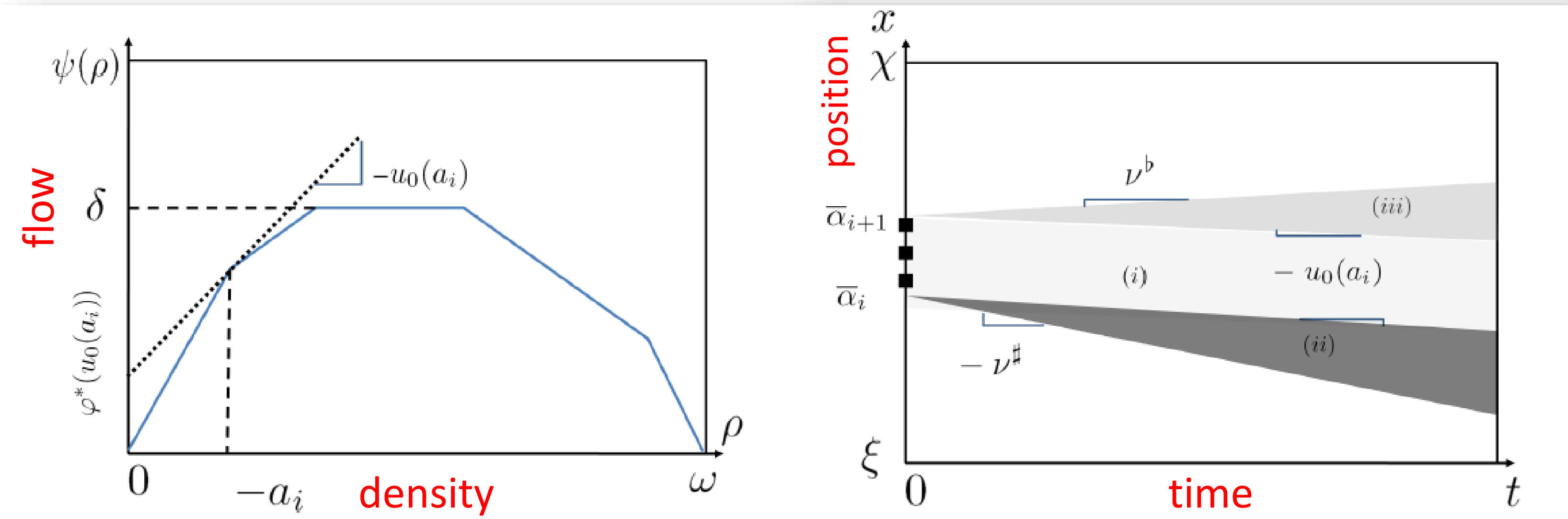
$$u_0(a_i) \in -\partial_+ \psi(-a_i)$$

Data

Model

$$\mathbf{M}_{\mathcal{M}_0,i}(t,x) = \begin{cases} (i) & t\psi(-a_i) + a_i x + b_i \\ & \text{if } u_0(a_i) \in \left[\frac{\overline{\alpha}_i - x}{t}, \frac{\overline{\alpha}_{i+1} - x}{t}\right] \\ (ii) & a_i \overline{\alpha}_i + b_i + t\varphi^*\left(\frac{\overline{\alpha}_i - x}{t}\right) \\ & \text{if } u_0(a_i) \le \frac{\overline{\alpha}_i - x}{t} \\ (iii) & a_i \overline{\alpha}_{i+1} + b_i + t\varphi^*\left(\frac{\overline{\alpha}_{i+1} - x}{t}\right) \\ & \text{if } u_0(a_i) \ge \frac{\overline{\alpha}_{i+1} - x}{t} \end{cases}$$



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The Lax-Hopf algorithm

Inf-morphism property

Let c(., .) be a function representing the initial conditions and boundary conditions

$$\forall(t,x) \in [0, t_{max}] \times [\xi, \chi], \quad \boldsymbol{c}(t,x) := \min_{j \in J} \boldsymbol{c}_j(t,x)$$

The solution is the minimum of partial solution components

$$\forall(t,x) \in [0, t_{max}] \times [\xi, \chi], \quad \boldsymbol{M_c}(t,x) := \min_{j \in J} \boldsymbol{M_{c_j}}(t,x)$$

Semi analytical property

If $c_j(\cdot,\cdot)$ is linear, the function $M_{c_j}(\cdot,\cdot)$ can be computed analytically

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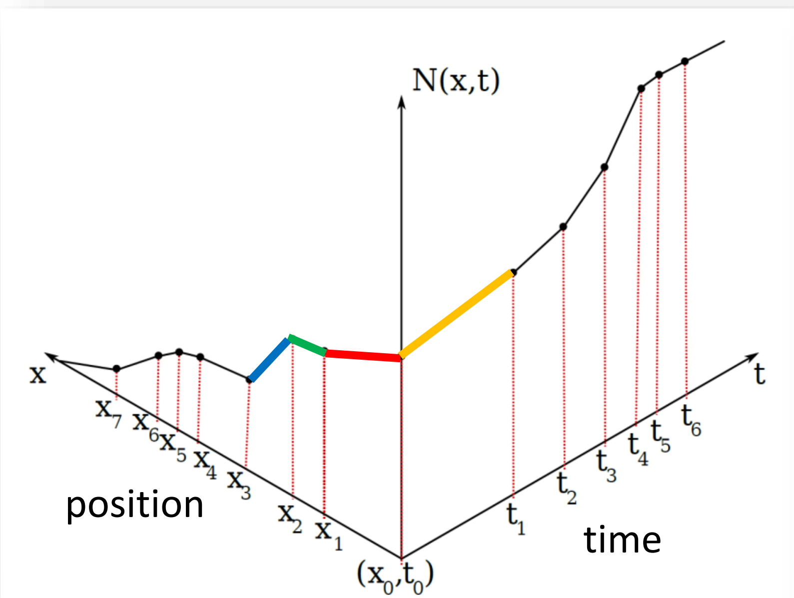
If $c_j(\cdot,\cdot)$ is linear, the function $M_{c_j}(\cdot,\cdot)$ can be computed analytically

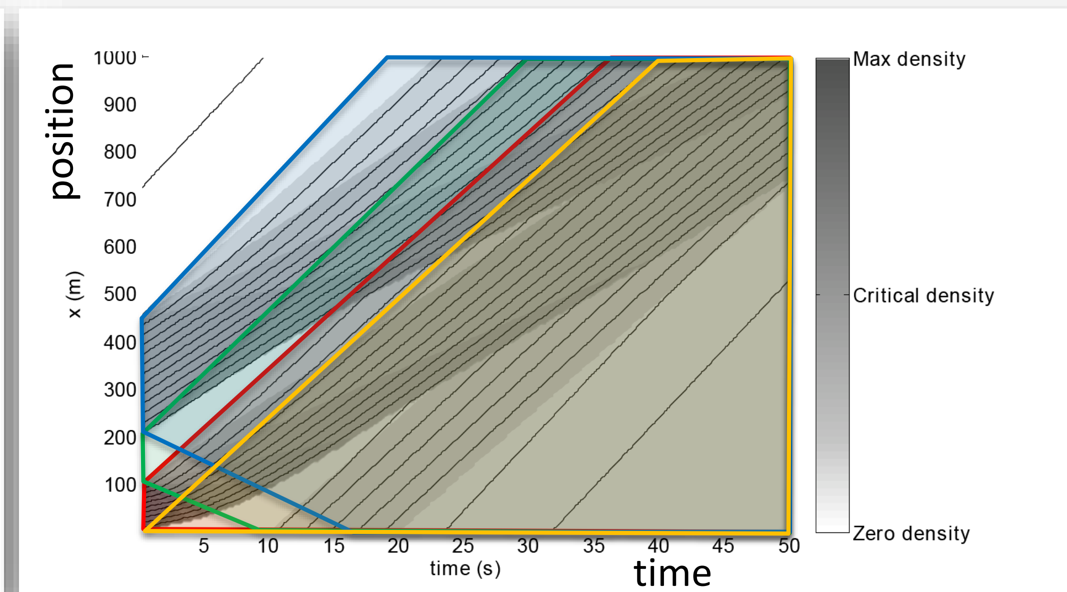
Illustration of the Lax-Hopf algorithm

$$\forall (t,x) \in [0, t_{\max}] \times [x, c], c(t,x) := \min_{j \in J} c_j(t,x)$$

The solution associated with the above boundary data function can be decomposed as:

$$\forall (t,x) \in [0, t_{\max}] \times [x, c], M_C(t,x) = \min_{j \in J} M_{c_j}(t,x)$$

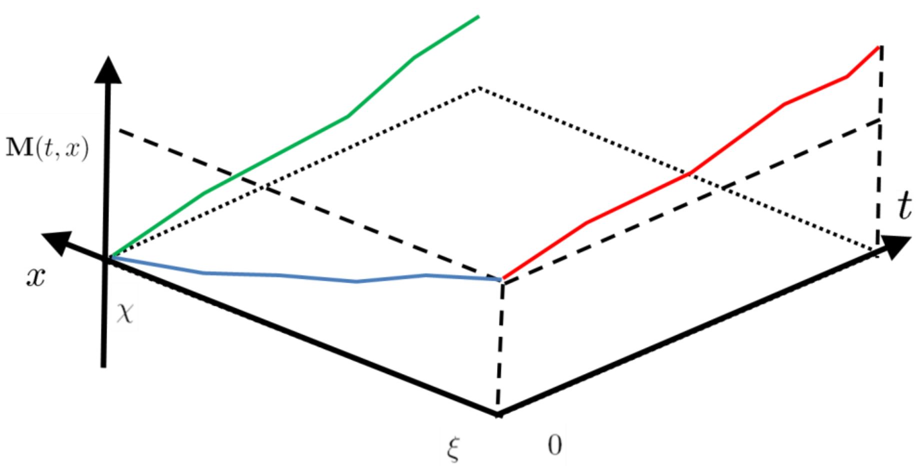




Compatibility conditions for existence of strong solutions

Compatibility conditions: $M_{c_j}(t,x) \geq c_i(t,x), \quad \forall (t,x) \in Dom(c_i), \forall (i,j) \in J^2$

Compatibility conditions with initial, upstream and downstream boundary conditions



- Initial condition
- Upstream boundary condition
- Downstream boundary condition

$$\begin{cases} M_{M_k}(0,x_p) \geq M_p(0,x_p) & \forall (k,p) \in K^2 \\ M_{M_k}(pT,\chi) \geq \beta_p(pT,\chi) & \forall k \in K, \quad \forall p \in N \\ M_{M_k}(\frac{\chi - x_k}{v_f},\chi) \geq \beta_p(\frac{\chi - x_k}{v_f},\chi) & \forall k \in K, \quad \forall p \in N \\ \qquad \text{s.t.} \quad \frac{\chi - x_k}{v_f} & \in [(p-1)T, pT] \\ M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall k \in K, \quad \forall p \in N \\ M_{M_k}(\frac{\xi - x_{k-1}}{w},\xi) \geq \gamma_p(\frac{\xi - x_{k-1}}{w},\xi) & \forall k \in K, \quad \forall p \in N \\ \qquad \text{s.t.} \quad \frac{\xi - x_{k-1}}{w} & \in [(p-1)T, pT] \end{cases} \tag{17}$$

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Compatibility conditions with initial, upstream and downstream boundary conditions

Linear in boundary flows

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Piecewise linear (concave) in boundary flows

$$\begin{cases} M_{\gamma_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in N^2 \\ M_{\gamma_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in N^2 \\ M_{\gamma_n}(nT + \frac{\chi - \xi}{v_f},\chi) \geq \beta_p(nT + \frac{\chi - \xi}{v_f},\chi) & \forall (n,p) \in N^2 \\ \quad \text{s.t.} \quad nT + \frac{\chi - \xi}{v_f} & \in [(p-1)T, pT] \end{cases} \quad (18)$$

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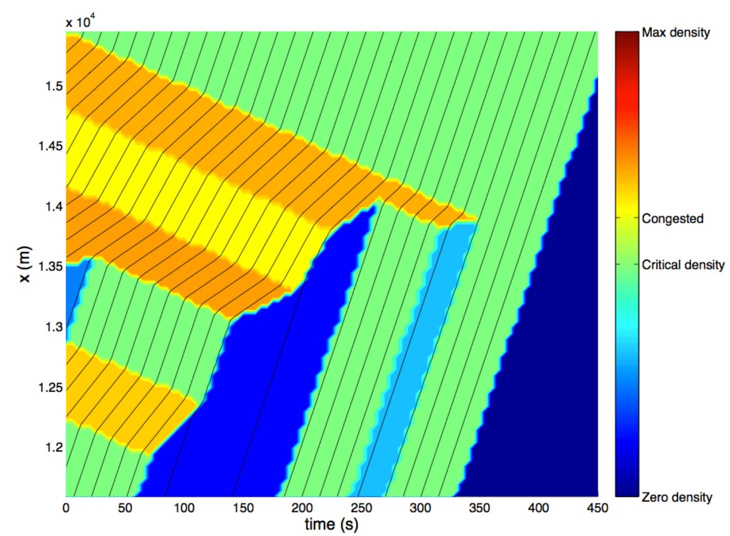
Optimization formulation of the boundary control problem

- Let x be the vector of boundary flows. The compatibility conditions imply that $Ax \leq b$
- Hence, if the objective function is linear in the boundary flows (e.g. when maximizing outflows, or minimizing vehicle accumulation over a link), the optimal boundary control problem becomes a LP:

$$\min c^T x$$
$$\text{s.t. } Ax \leq b$$

- The initial conditions only influence the right hand side vector b. The model parameters (ψ) influence both A and b



[Li Canepa Claudel (2014)]

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Stochastic formulation

- In actual control problems initial condition and/or the model parameters may be uncertain
- This can dramatically affect the results, since the solution to the control problem may lead to worse congestion (when applied to a real traffic scenario) than no control at all
- We assume that the initial condition has Gaussian uncertainty
  Solution: use chance constrained-optimization (uncertainty appears only in the constraints)

$$M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi), \quad \forall k \in K, \quad \forall p \in N$$

⇩

$$P(M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)) \geq 1-\alpha, \quad \forall k \in K, \quad \forall p \in N$$

Stochastic model

How to convert the chance constraint:

$$P(M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)) \geq 1-\alpha, \quad \forall k \in K, \quad \forall p \in N$$

into a deterministic (linear) expression?

Assume the uncertainty is normally distributed $\rho_k \sim n(\rho_k, \sigma_k)$

$$\rho_c \leq \rho_k + z_{1-\alpha}\sigma_k$$

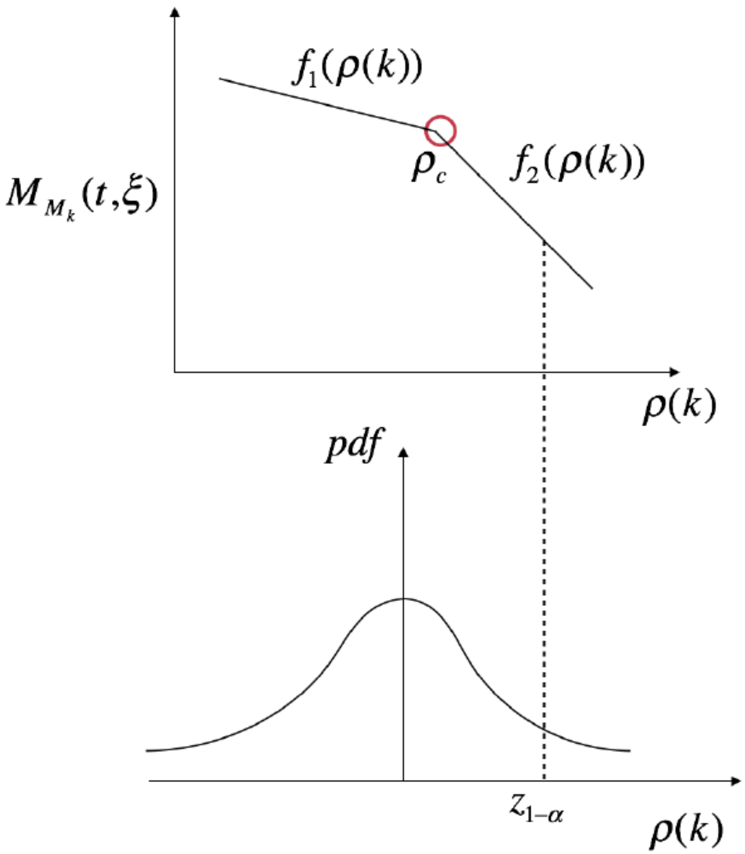
$$P(M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)) \geq 1-\alpha \Leftrightarrow f_2(\rho_k + z_{1-\alpha}\sigma_k) \geq \gamma_p(pT,\xi)$$

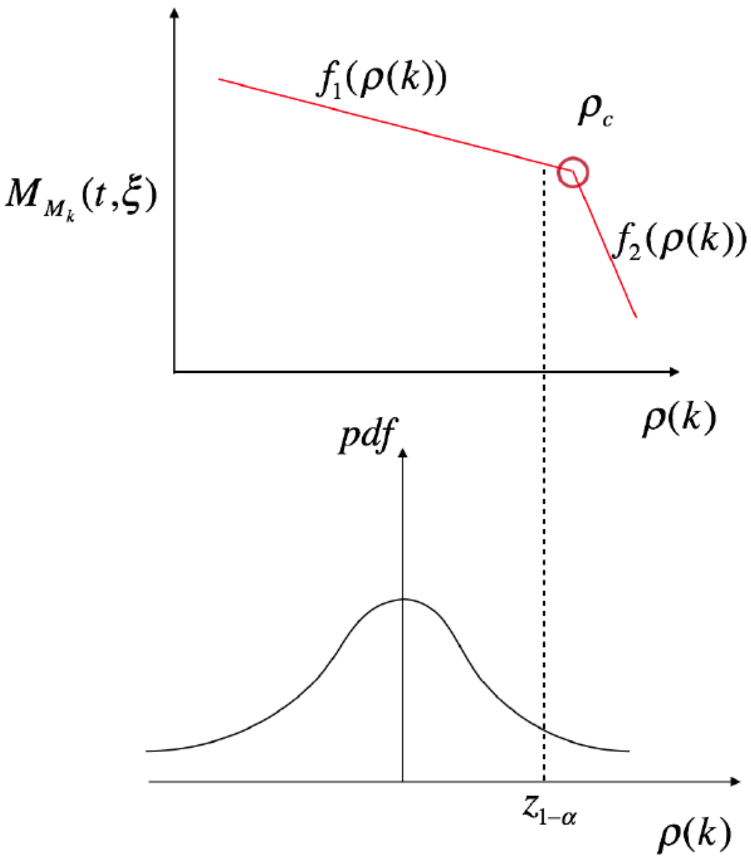
If $\rho_c \geq \rho_k + z_{1-\alpha}\sigma_k$,

$$P(M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)) \geq 1-\alpha \Leftrightarrow f_1(\rho_k + z_{1-\alpha}\sigma_k) \geq \gamma_p(pT,\xi)$$

Where $f_1(\cdot)$ and $f_2(\cdot)$ are linear

Stochastic model





Stochastic model

Chance constraints can be equivalently reformulated as linear constraints

$$P(M_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi)) \geq 1-\alpha \Leftrightarrow \begin{cases} -\sum_{i=1}^{k-1} \rho(i)x + \rho_c(pTv_f + (k-1)x - \xi) \geq \sum_{i=1}^{p} q_{in}(i)T, \\ \text{if } t \geq \frac{\xi-(k-1)X}{w}, \quad \text{and } \rho_k + z_{1-\alpha}\sigma_k \leq \rho_c \\ -\sum_{i=1}^{k-1} \rho(i)X + (\rho(k) + z_{1-\alpha}\sigma_k)(tw + (k-1)X - \xi) - \rho_m tw \\ \geq \sum_{i=1}^{p} q_{in}(i)T, \\ \text{if } \frac{\xi-(k-1)X}{w} \leq t \leq \frac{\xi-kX}{w}, \quad \text{and } \rho_k + z_{1-\alpha}\sigma_k \geq \rho_c \\ -\sum_{i=1}^{k-1} \rho(i)X - (\rho(k) + z_{1-\alpha}\sigma_k)X + \rho_c(tw + kX - \xi) - \rho_m tw \\ \geq \sum_{i=1}^{p} q_{in}(i)T, \\ \text{if } t \geq \frac{\xi-kX}{w}, \quad \text{and } \rho_k + z_{1-\alpha}\sigma_k \geq \rho_c \end{cases}$$

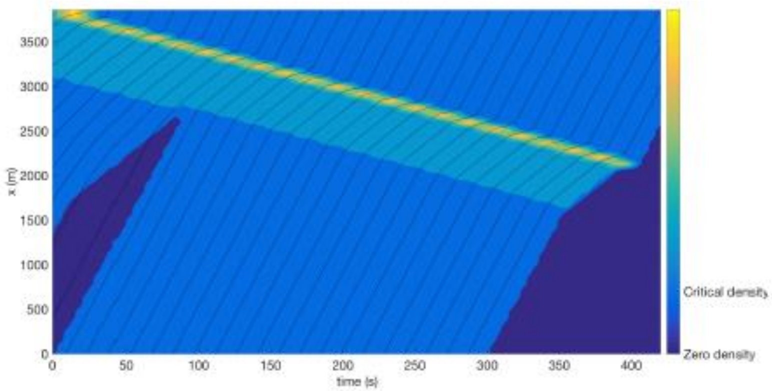
Can be extended to other probability functions (only requires the knowledge of the c.d.f.)

Example simulation

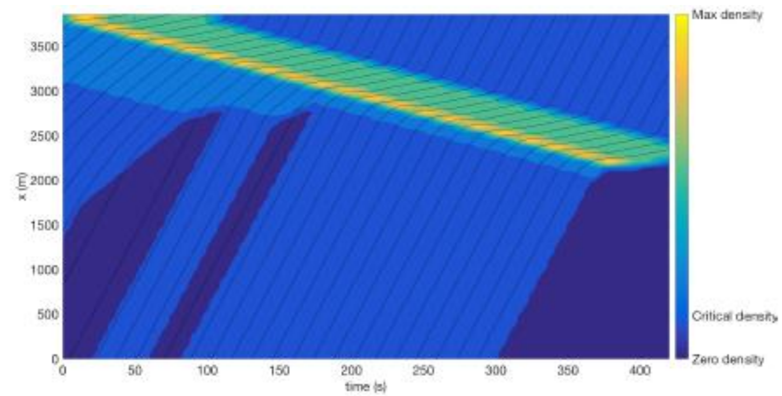
- Highway link (I 880, CA) of 3.8 km with 7 segments of constant initial density
- Simulation time horizon of 7 minutes (28 time steps)
- Model parameters:
  - Critical density: 0.03 /m;
  - Free flow speed: 30 m/s;
  - Jam density: 0.24 /m
- Means of initial densities are drawn in the range [0.01, 0.07];
- Four scenarios with $\sigma \in \{0.01, 0.02, 0.03, 0.04\}$ are tested
- Confidence level $1 - \alpha = 0.975$

Example simulation

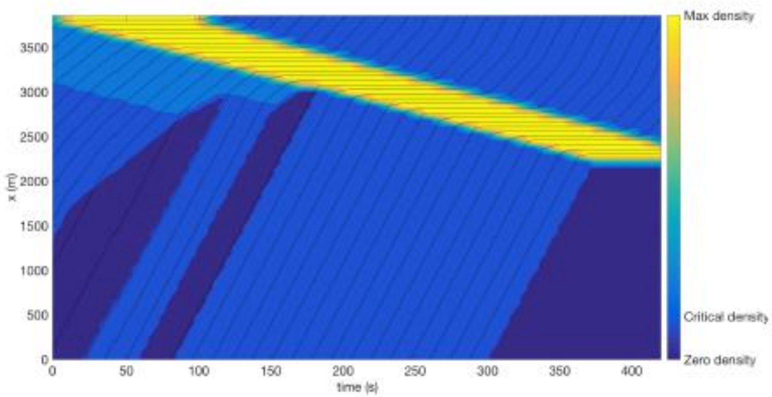
- Objective function: maximize total throughput



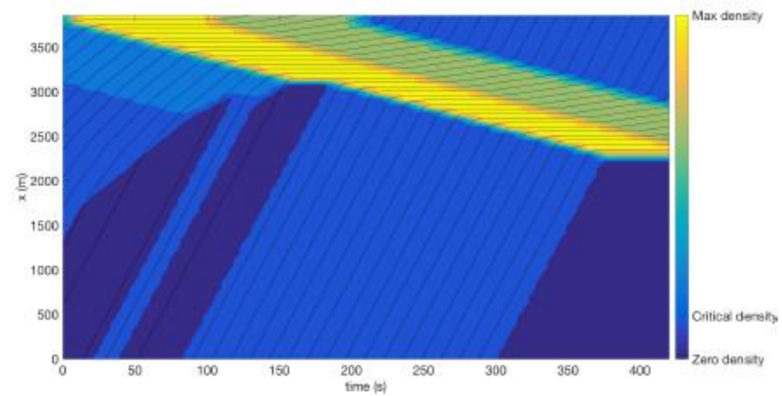
(a) $\sigma = 0.01$



(b) $\sigma = 0.02$



(c) $\sigma = 0.03$



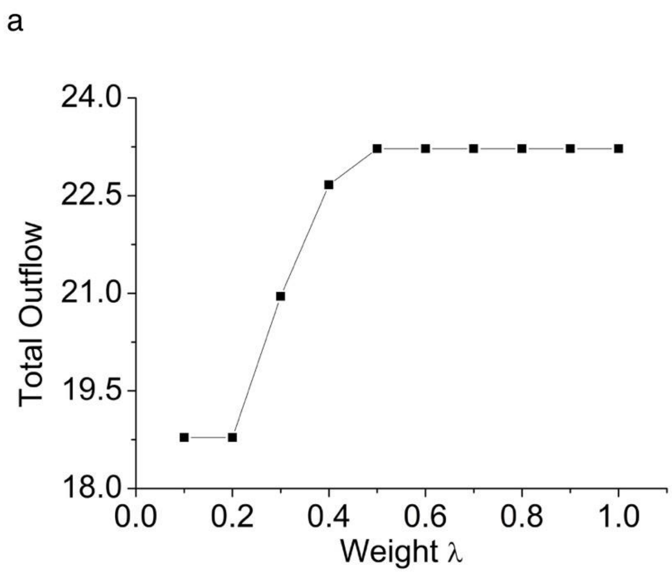
(d) $\sigma = 0.024$

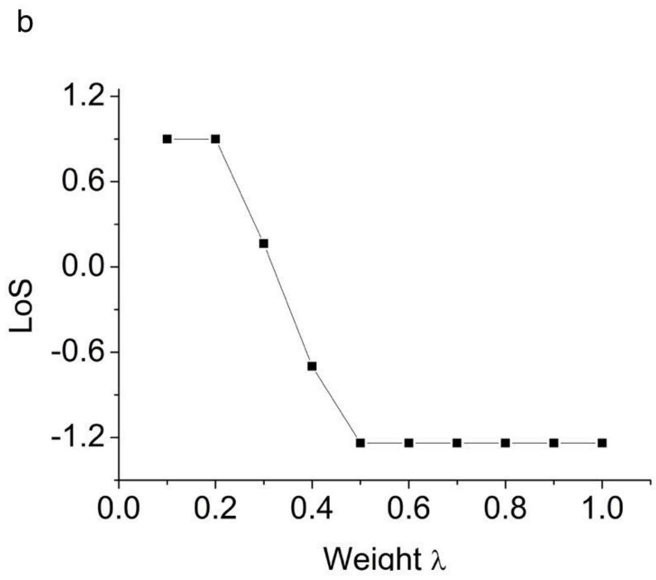
Example simulation

- Dual objective: maximize throughput and minimize accumulation

$$min \quad -\lambda \sum_{i=1}^{n_{max}} q_{out}(i) + (1-\lambda)Q$$

$$s.t. \quad Q \geq \sum_{j=1}^{i} (q_{in}(j) - q_{out}(j)), \quad \forall i \in N$$





Outline

Motivation

Traffic flow control of highways using Hamilton-Jacobi equations
- Background
- Problem definition
- Optimization formulations

Robust flow control problems on single links using chance constraints
- Initial condition uncertainty

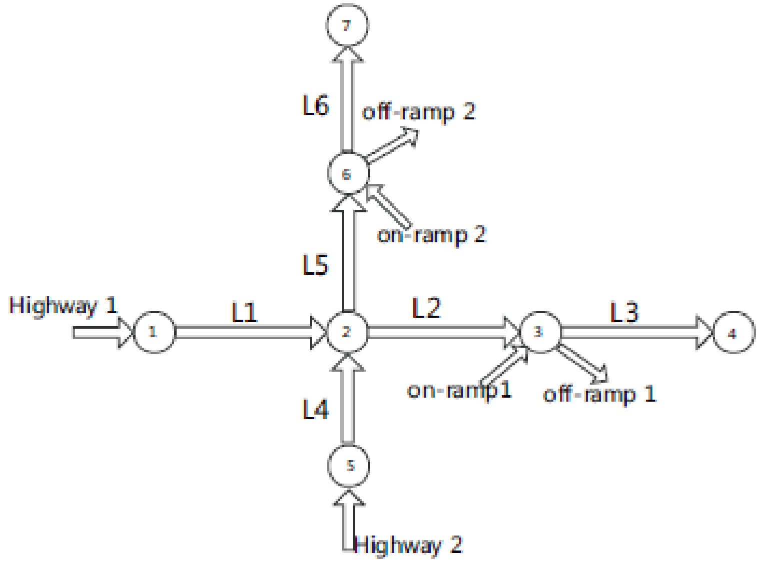
Extension to network problems
- Problem formulation
- Simulation results

Conclusion

Network problem formulation

- Robust network boundary control problems can be handled similarly, provided that every link is controlled (otherwise robust MILP)

$$
\begin{aligned}
min \quad & -\sum_{i=1}^{n_{max}}\sum_{j=1}^{n_l}(h(q_{out}(i,j)+q_{in}(i,j))- \\
& \eta(q_{in}(i,j)-q_{out}(i,j))-q_d^{out}(i,j)-q_d^{in}(i,j)-y(i)) \\
s.t. \quad & q_d^{out}(i,j)\geq q_{out}(i,j)-q_{out}(i-1,j), \quad \forall i\geq 2, \quad j\in L \\
& q_d^{out}(i,j)\geq q_{out}(i-1,j)-q_{out}(i,j), \quad \forall i\geq 2, \quad j\in L \\
& q_d^{in}(i,j)\geq q_{in}(i,j)-q_{in}(i-1,j), \quad \forall i\geq 2, \quad j\in L \\
& q_d^{in}(i,j)\geq q_{in}(i-1,j)-q_{in}(i,j), \quad \forall i\geq 2, \quad j\in L \\
& y(i)\geq n_{lane}(4)q_{out}(i,1)-n_{lane}(1)q_{out}(i,4), \quad \forall i \\
& y(i)\geq n_{lane}(1)q_{out}(i,4)-n_{lane}(4)q_{out}(i,1), \quad \forall i \\
& q_{on}(i,1)\geq q_{out}(i,2)/n_{lane}(2), \quad \forall i\in N \\
& q_{on}(i,2)\geq q_{out}(i,4)/n_{lane}(4), \quad \forall i\in N \\
& q_{out}(i,3)\leq \psi'(\rho_3), \quad \forall i\in N \\
& q_{out}(i,6)\leq \psi'(\rho_6), \quad \forall i\in N \\
& (35)-(39), \quad \forall j\in L \\
& (27), \quad \forall v\in V \\
& q_d^{out}(i,j)\geq 0, \quad q_d^{in}(i,j)\geq 0 \quad \forall i, \quad j \\
& q_{out}(i,j)\geq 0, \quad q_{in}(i,j)\geq 0 \quad \forall i, \quad j
\end{aligned}
$$



$$
\begin{bmatrix} q_{out} \\ q_{off} \end{bmatrix} = \begin{bmatrix} P^1 & P^2 \\ P^3 & 0 \end{bmatrix} \begin{bmatrix} q_{in} \\ q_{on} \end{bmatrix}, \qquad (27)
$$

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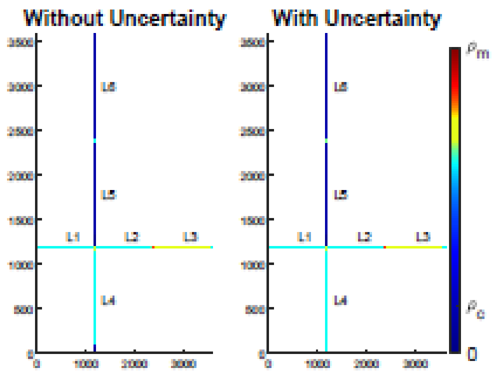
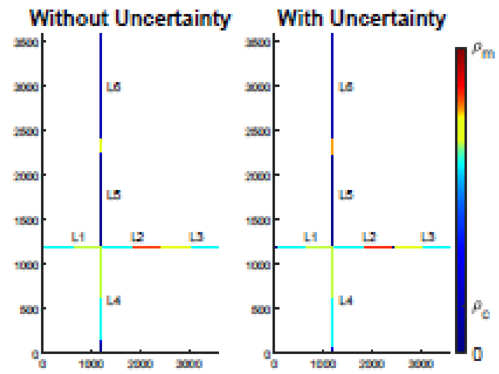
Simulation results

- Optimal boundary flows evaluated on the worst-case initial condition

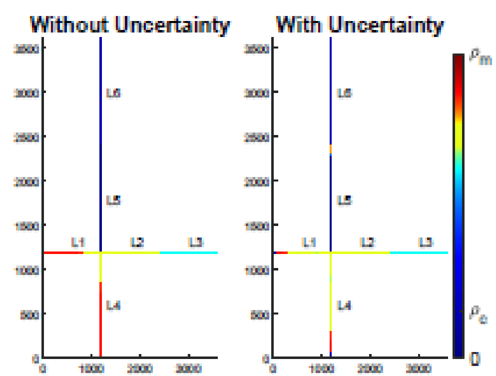


(a) $t = 0$



(b) $t = 100$

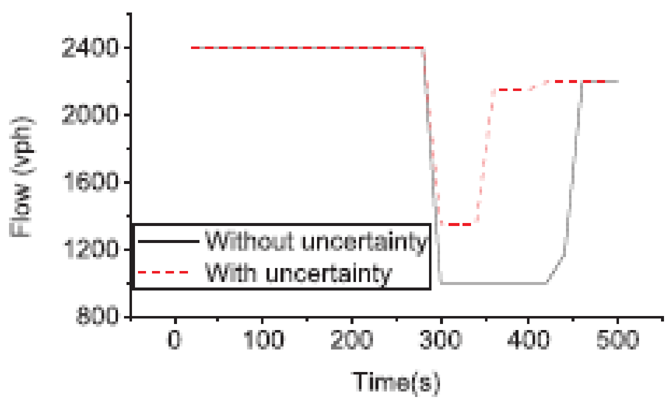
(e) $t = 400$



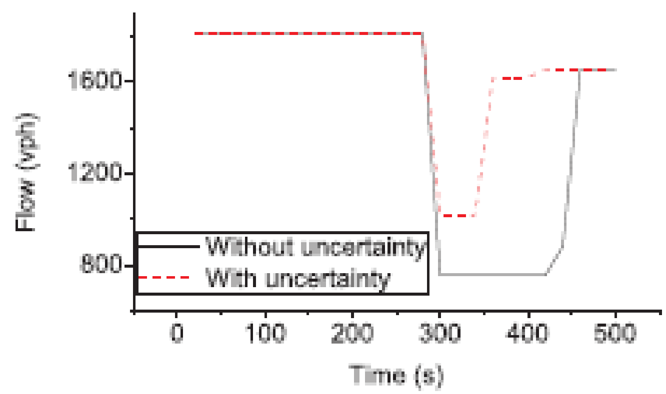
(f) $t = 500$

Simulation results

- Optimal boundary flows evaluated on the worst-case initial condition



(a) link 1



(b) link 4

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Conclusion and future work

- Boundary control problems (single link, non robust) can be formulated as linear programs. Network problems (non robust) are also linear programs (if all links are controlled)
- Initial condition uncertainty can be modeled as chance constraints
- Limitations (future work)
  - integrating joint chance constraints if the mode (congested/uncongested) of each initial condition block is known
  - Combine speed-boundary control and investigate the corresponding robust control problem
  - Investigate the robust control problem with model uncertainty

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