Workshop "Resilient Control of Infrastructure Networks"

Macroscopic traffic flow models on road networks

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Outline of the talk

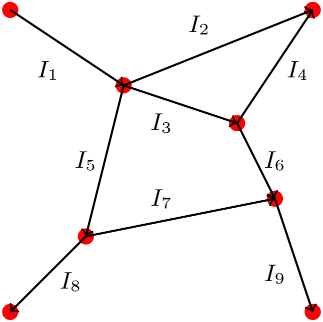
1. Traffic flow on networks
2. The Riemann Problem at point junctions
3. Existence of solutions
4. Examples
5. The Riemann Problem at junctions with buffer
6. The Riemann Problem at junctions for 2nd order models
7. A multi-class model on networks
8. Examples

Outline of the talk

Conservation laws on networks[1]

Networks

Finite collection of directed arcs $I_i =]a_i, b_i[$ connected by nodes



[1] [Holden-Risebro 1995; Garavello-Piccoli 2006]

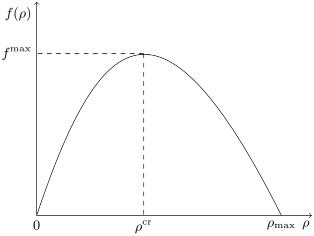
LWR model[2]

Non-linear transport equation: PDE for mass conservation

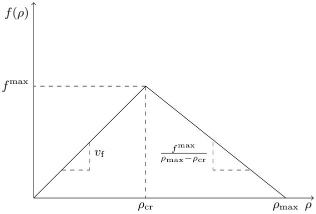
$$\partial_t \rho + \partial_x f(\rho) = 0 \qquad x \in \mathbb{R}, t > 0$$

- $\rho \in [0, \rho_{\max}]$ mean traffic density
- $f(\rho) = \rho v(\rho)$ flux function

Empirical flux-density relation: fundamental diagram



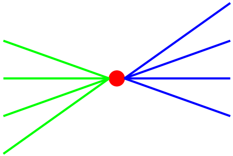
Greenshields '35



Newell-Daganzo

[2] [Lighthill-Whitham 1955, Richards 1956]

Extension to networks



m incoming arcs

n outgoing arcs

junction

- LWR on networks:
  [Holden-Risebro, 1995; Coclite-Garavello-Piccoli, 2005; Garavello-Piccoli, 2006]
  - LWR on each road
  - Optimization problem at the junction

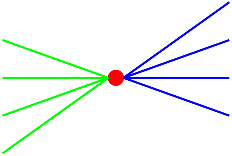
- Modeling of junctions with a buffer:
  [Herty-Lebacque-Moutari, 2009; Garavello-Goatin, 2012; Garavello, 2014; Bressan-Nguyen, 2015; LaurentBrouty&al, 2019]
  - Junction described by one or more buffers
  - Suitable for optimization and Nash equilibrium problems

Outline of the talk

Riemann problem at J

$$\begin{cases} \partial_t \rho_k + \partial_x f(\rho_k) = 0 \\ \rho_k(0,x) = \rho_{k,0} \end{cases}$$

$$k = 1, \dots, n+m$$



Riemann solver: $\mathcal{RS}_J : (\rho_{1,0}, \dots, \rho_{n+m,0}) \longmapsto (\bar\rho_1, \dots, \bar\rho_{n+m})$ s.t.

- conservation of cars: $\sum_{i=1}^{n} f_i(\bar\rho_i) = \sum_{j=n+1}^{n+m} f_j(\bar\rho_j)$
- waves with negative speed in incoming roads
- waves with positive speed in outgoing roads

Consistency condition:

$$\mathcal{RS}_J\big(\mathcal{RS}_J(\rho_{1,0}, \dots, \rho_{n+m,0})\big) = \mathcal{RS}_J(\rho_{1,0}, \dots, \rho_{n+m,0}) \qquad \textbf{(CC)}$$

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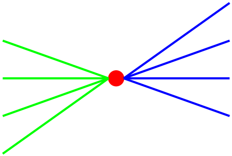

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Set $\bar\gamma_k = f_k(\bar\rho_k)$

Dynamics at junctions

(A) **prescribe a fixed distribution of traffic in outgoing roads**

$$A = \{a_{ji}\} \in \mathbb{R}^{m \times n} : \ 0 < a_{ji} < 1, \sum_{j=n+1}^{n+m} a_{ji} = 1$$

outgoing fluxes $= A\cdot$ incoming fluxes
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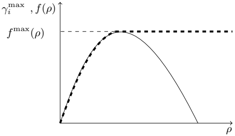
(A)+(B) equivalent to a LP optimization problem and a unique solution to RPs

More incoming than outgoing roads $\Longrightarrow$ priority parameters

Demand & Supply [3]

Incoming roads $i = 1, \dots, n$:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}) & \text{if } 0 \le \rho_{i,0} < \rho^{\text{cr}} \\ f^{\max} & \text{if } \rho^{\text{cr}} \le \rho_{i,0} \le 1 \end{cases}$$



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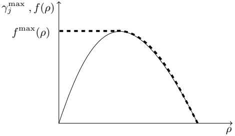

[3] [Lebacque, 1996]

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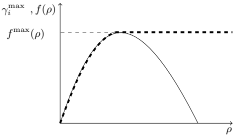

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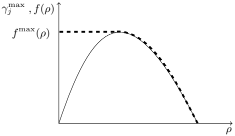


Admissible fluxes at junction: $\Omega_l = [0, \gamma_l^{\max}]$

[3] [Lebacque, 1996]

Priority Riemann Solver[4]

(A) distribution matrix of traffic from incoming to outgoing roads

$$A = \{a_{ji}\} \in \mathbb{R}^{m \times n}: \quad 0 \le a_{ji} \le 1, \sum_{j=n+1}^{n+m} a_{ji} = 1$$

(B) priority vector

$$P = (p_1, \dots, p_n) \in \mathbb{R}^n: \quad p_i > 0, \sum_{i=1}^{n} p_i = 1$$

(C) feasible set

$$\Omega = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}$$

[4] [DelleMonache-Goatin-Piccoli, CMS 2018]

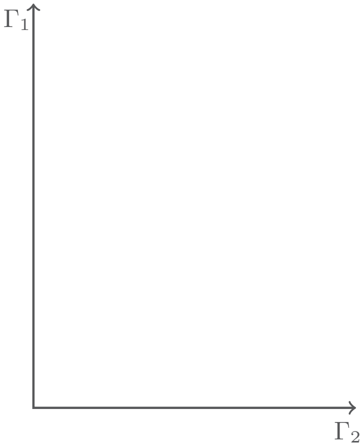
Priority Riemann Solver

Algorithm 1 Recursive definition of $\mathcal{PRS}$

Set $J = \emptyset$ and $J^c = \{1, \dots, n\} \setminus J$.
while $|J| < n$ **do**
  $\forall i \in J^c \rightarrow h_i = \max\{h : h\, p_i \leq \gamma_i^{max}\} = \frac{\gamma_i^{max}}{p_i}$,
  $\forall j \in \{n+1 \dots, n+m\} \rightarrow h_j = \sup\{h : \sum_{i \in J} a_{ji} Q_i + h(\sum_{i \in J^c} a_{ji} p_i) \leq \gamma_j^{max}\}$.
  Set $\hbar = \min_{ij}\{h_i, h_j\}$.
  if $\exists\ j$ s.t. $h_j = \hbar$ **then**
    Set $Q = \hbar\, P$ and $J = \{1, \dots, n\}$.
  else
    Set $I = \{i \in J^c : h_i = \hbar\}$ and $Q_i = \hbar\, p_i$ for $i \in I$.
    Set $J = J \cup I$.
  end if
end while

$\mathcal{PRS}$ in practice

2×2 junction ($n = 2$, $m = 2$):



1. Define the spaces of the incoming fluxes

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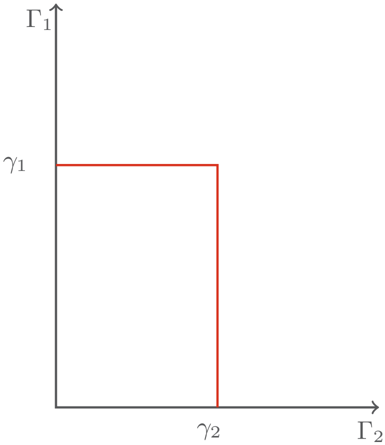

1. Define the spaces of the incoming fluxes
2. Consider the demands

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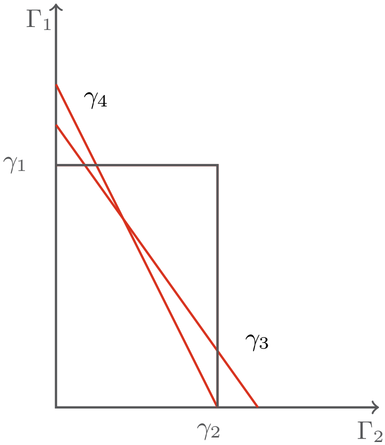

1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply lines

$\mathcal{PRS}$ in practice

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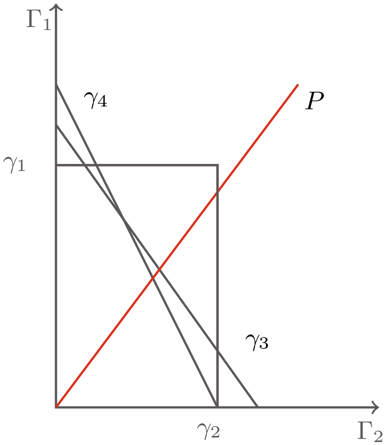

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4. Trace the priority line

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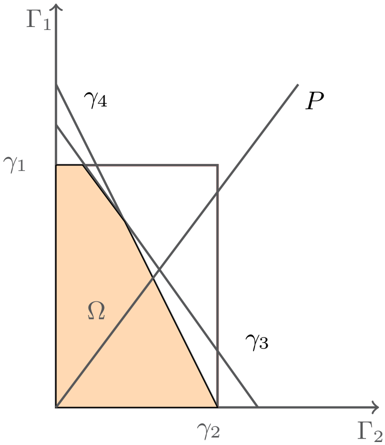

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4. Trace the priority line
5. The feasible set is given by Ω

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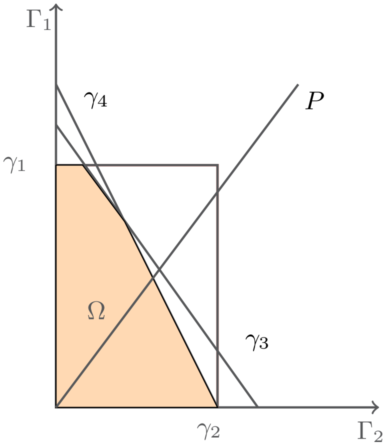

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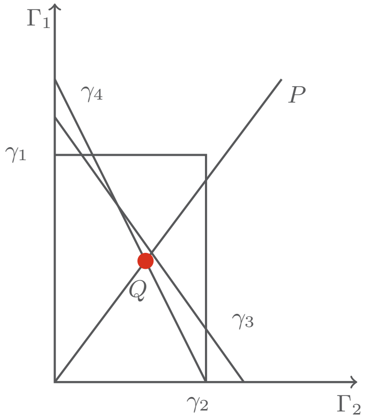
Different situations can occur

$\mathcal{PRS}$: optimal point

P intersects the supply lines inside Ω



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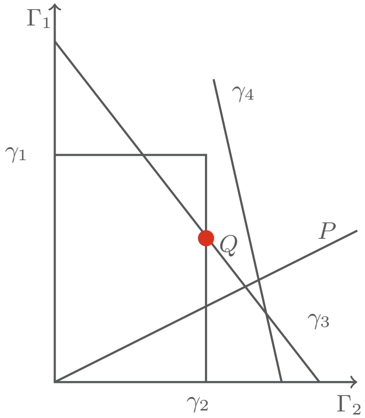

$\mathcal{PRS}$

Definition ($\mathcal{PRS}$)

$Q = (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ incoming fluxes defined by Algorithm 1
$A \cdot Q^T = (\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})^T$ outgoing fluxes
Set

$$\bar{\rho}_i = \begin{cases} \rho_{i,0} & \text{if } f(\rho_{i,0}) = \bar{\gamma}_i \\ \rho \geq \rho^{\text{cr}} & \text{s.t. } f(\rho) = \bar{\gamma}_i \end{cases} \qquad i \in \{1, \ldots, n\}$$

$$\bar{\rho}_i = \begin{cases} \rho_{j,0} & \text{if } f(\rho_{j,0}) = \bar{\gamma}_i \\ \rho \leq \rho^{\text{cr}} & \text{s.t. } f(\rho) = \bar{\gamma}_j \end{cases} \qquad j \in \{n+1, \ldots, n+m\}$$

Then, $\mathcal{PRS} : [0, \rho_{\max}]^{n+m} \to [0, \rho_{\max}]^{n+m}$ is given by

$$\mathcal{PRS}(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}) \,.$$

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Remark: $\mathcal{PRS}$ may be obtained as limit of solvers defined by Dynamic Traffic Assignment based on junctions with queues
[Bressan-Nordli, NHM, to appear]

Outline of the talk

Riemann solver: properties[5]

Definition (P1)

$$\mathcal{RS}(\rho_{1,0},\dots,\rho_{n+m,0}) = \mathcal{RS}(\rho'_{1,0},\dots,\rho'_{n+m,0})$$

if $\rho_{l,0} = \rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum ($\gamma_l^{\max} \neq f^{\max}$).

Definition (P2)

$$\Delta \mathrm{TV}_f(\bar t) \le C \min\left\{|f(\rho_{l,0}) - f(\rho_l)|, |\Gamma(\bar t+) - \Gamma(\bar t-)| + \left|\bar h(\bar t+) - \bar h(\bar t-)\right|\right\}$$
$$\Delta \bar h(\bar t) \le C\,|f(\rho_{l,0}) - f(\rho_l)|$$

with $C \ge 1$, where $\Gamma(t) := \sum_{i=1}^n f(\rho_i(t,0-))$, $\bar h = \sup\{h \in \mathbb{R}^+ : hP \in \Omega\}$.

Definition (P3)

If $f(\rho_l) < f(\rho_{l,0})$: $\quad \Delta\Gamma(\bar t) \le C\left|\bar h(\bar t+) - \bar h(\bar t-)\right|, \quad \bar h(\bar t+) \le \bar h(\bar t-)$.

[5] Garavello-Piccoli, AnnIHP 2009

Cauchy problem: existence results

Theorem (DelleMonache-Goatin-Piccoli, CMS 2018)

If a Riemann solver satisfies (P1)-(P3), then every Cauchy problem with BV initial data admits a weak solution.

Proof: Wave-Front Tracking, bound on $\mathrm{TV}(f)$ and “big shocks”.

Proposition (DelleMonache-Goatin-Piccoli, CMS 2018)

The Priority Riemann Solver $\mathcal{PRS}$ satisfies (P1)-(P3) for junctions with $n \leq 2$, $m \leq 2$ and $0 < a_{ji} < 1$ for all i, j.

Cauchy problem: counterexample for Lipschitz dependence

Proposition (Garavello-Piccoli, Section 5.4)

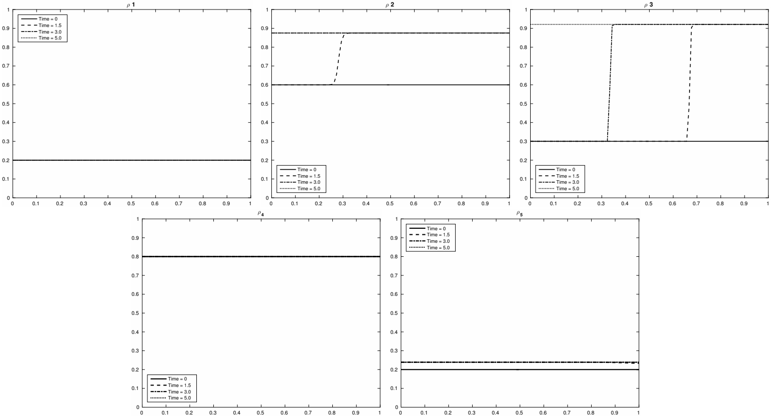
Let $C > 0$ and a 2×2 junction with $\mathcal{RS}_J(\rho_{1,0}, \ldots, \rho_{4,0}) = (\rho_{1,0}, \ldots, \rho_{4,0})$. Then there exist two piece-wise constant initial data such that the $\mathbf{L^1}$-distance between the corresponding solutions increases by C

$$\|\rho(t, \cdot) - \bar{\rho}(t, \cdot)\|_1 \geq C\|\rho_0 - \bar{\rho}_0\|_1$$

Outline of the talk

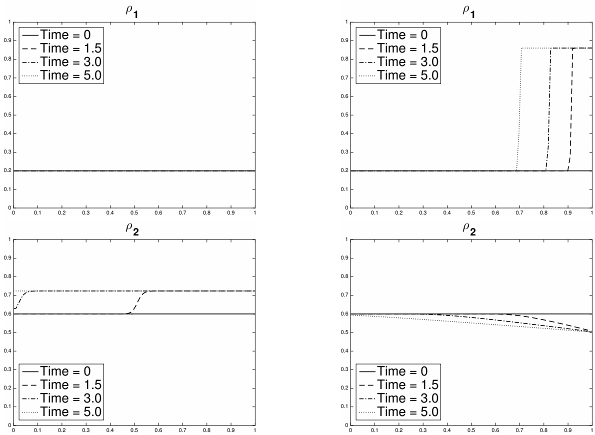
$\mathcal{PRS}$ on 3×2 junction

$$A = \begin{bmatrix} 0.5 & 0.6 & 0.2 \\ 0.5 & 0.4 & 0.8 \end{bmatrix} \quad P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \end{bmatrix}$$



$\mathcal{PRS}$ VS $\mathcal{RS}_{\mathrm{CGP}}$ on 2×2 junction

$$A = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix} \quad P = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \quad \begin{matrix} \rho_{1,0} = 0.2 & \rho_{3,0} = 0.3 \\ \rho_{2,0} = 0.6 & \rho_{4,0} = 0.8 \end{matrix}$$



$\mathcal{PRS}$ $\qquad$ $\mathcal{RS}_{\mathrm{CGP}}$

In summary

General Riemann Solver at junctions:

- no restriction on A
- no restriction on the number of roads
- priorities come before flux maximization
- compact algorithm to compute solutions
- general existence result

Basic bibliography

- H. Holden, N. Risebro. A mathematical model of traffic flow on a network of unidirectional roads. SIAM J. Math. Anal. 1995.
- M. Garavello, B. Piccoli. Traffic flow on networks. AIMS Series on Applied Mathematics, 1. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.
- M. Garavello, K. Han, B. Piccoli. Models for Vehicular Traffic on Networks. AIMS Series on Applied Mathematics, 9, American Institute of Mathematical Sciences(AIMS), Springfield, MO, 2016.
- M.L. Delle Monache, P. Goatin, B. Piccoli. Priority-based Riemann solver for traffic flow on networks. Comm. Math. Sci. 2018.

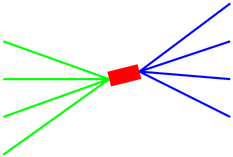
Outline of the talk

Riemann problem at J with buffer

$$\begin{aligned} &\partial_t \rho_k + \partial_x f(\rho_k) = 0 \\ &\rho_k(0,x) = \rho_{k,0} \end{aligned} \qquad k = 1, \ldots, n+m$$

$$r'(t) = \sum_{i=1}^{n} f(\rho_i(t,0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t,0+))$$

$r(0) = r_0 \in [0, r_{max}]$ buffer load



Riemann solver:

$\mathcal{RS}_{r(t)} : (\rho_{1,0}, \ldots, \rho_{n+m,0}, r_0) \longmapsto (\rho_1(t,x), \ldots, \rho_{n+m}(t,x), r(t))$ s.t.

- buffer dynamics: $r'(t) = \sum_{i=1}^{n} f_i(\rho_i(t,0-)) - \sum_{j=n+1}^{n+m} f_j(\rho_j(t,0+))$
- waves with negative speed in incoming roads
- waves with positive speed in outgoing roads

Consistency condition:

$$\mathcal{RS}_{\bar r}\big(\mathcal{RS}_{\bar r}(\rho_{1,0}, \ldots, \rho_{n+m,0})\big) = \mathcal{RS}_{\bar r}(\rho_{1,0}, \ldots, \rho_{n+m,0}), \ \forall \bar r \in [0, r_{max}]$$

Riemann problem with buffer: construction

Let $\theta_k \in]0,1[$, $k = 1, \ldots, n+m$, s.t. $\sum_{i=1}^{n} \theta_i = \sum_{j=n+1}^{n+m} \theta_j = 1$
$\mu \in]0, \max\{n,m\} f^{\max}[$: maximum load entering the junction

1. $\Gamma_{inc}^1 = \sum_{i=1}^{n} \gamma_i^{\max}$ $\quad\quad$ $\Gamma_{out}^1 = \sum_{j=n+1}^{n+m} \gamma_j^{\max}$

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2. $$\Gamma_{inc} = \begin{cases} \min\{\Gamma_{inc}^1, \mu\} \\ \min\{\Gamma_{inc}^1, \Gamma_{out}^1, \mu\} \end{cases} \qquad \Gamma_{out} = \begin{cases} \min\{\Gamma_{out}^1, \mu\} & \bar{r} \in [0, r_{max}[\\ \min\{\Gamma_{inc}^1, \Gamma_{out}^1, \mu\} & \bar{r} = r_{max} \end{cases}$$

Riemann problem with buffer: construction

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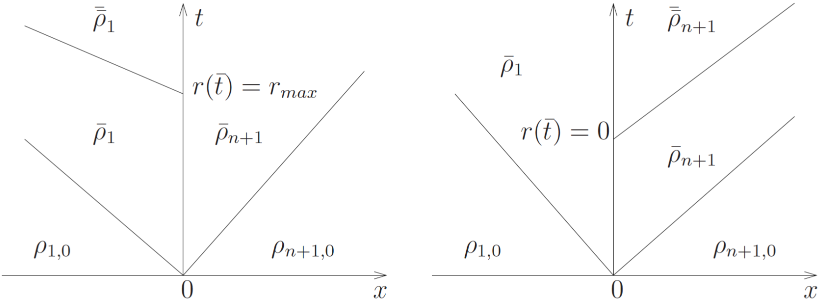
1. $\Gamma_{inc}^1 = \sum_{i=1}^{n} \gamma_i^{\max} \qquad \Gamma_{out}^1 = \sum_{j=n+1}^{n+m} \gamma_j^{\max}$

2. $$\Gamma_{inc} = \begin{cases} \min\{\Gamma_{inc}^1, \mu\} \\ \min\{\Gamma_{inc}^1, \Gamma_{out}^1, \mu\} \end{cases} \qquad \Gamma_{out} = \begin{cases} \min\{\Gamma_{out}^1, \mu\} & \bar{r} \in [0, r_{max}[\\ \min\{\Gamma_{inc}^1, \Gamma_{out}^1, \mu\} & \bar{r} = r_{max} \end{cases}$$

3. $(\bar{\gamma}_1, \dots, \bar{\gamma}_n) = \mathrm{Proj}_{I_{\Gamma_{inc}}} (\theta_1 \Gamma_{inc}, \dots, \theta_n \Gamma_{inc})$
$(\bar{\gamma}_{n+1}, \dots, \bar{\gamma}_{n+m}) = \mathrm{Proj}_{J_{\Gamma_{out}}} (\theta_{n+1} \Gamma_{inc}, \dots, \theta_{n+m} \Gamma_{inc})$
where
$I_{\Gamma_{inc}} = \{(\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^{n} [0, \gamma_i^{\max}] \colon \sum_{i=1}^{n} \gamma_i = \Gamma_{inc}\}$
$J_{\Gamma_{out}} = \left\{(\gamma_{n+1}, \dots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} [0, \gamma_j^{\max}] \colon \sum_{j=n+1}^{n+m} \gamma_j = \Gamma_{out}\right\}.$

Riemann problem with buffer: example



The solution to the Riemann problem when $n = m = 1$:
$\Gamma_{inc} > \Gamma_{out}$ on the left, $\Gamma_{inc} < \Gamma_{out}$ on the right.

Cauchy problem with buffer: existence

Theorem (Garavello-Goatin, DCDS-A 2012)

For every $T > 0$, the Cauchy problem admits a weak solution at J $(\rho_1, \ldots, \rho_{n+m}, r)$ such that

1. *for every $l \in \{1, \ldots, n+m\}$, ρ_l is a weak entropic solution of*
$$\partial_t \rho_l + \partial_x f(\rho_l) = 0$$
in $[0, T] \times I_l$;
2. *for every $l \in \{1, \ldots, n+m\}$, $\rho_l(0, x) = \rho_{0,l}(x)$ for a.e. $x \in I_l$;*
3. *for a.e. $t \in [0, T]$*
$$\mathcal{RS}_{r(t)}(\rho_1(t, 0-), \ldots, \rho_{n+m}(t, 0+)) = (\rho_1(t, 0-), \ldots, \rho_{n+m}(t, 0+));$$
4. *for a.e. $t \in [0, T]$*
$$r'(t) = \sum_{i=1}^{n} f(\rho_i(t, 0-)) - \sum_{j=n+1}^{n+m} f(\rho_j(t, 0+)).$$

Proof: Wave-Front Tracking, bound on $\mathrm{TV}(f)$ and “big shocks”.

Cauchy problem with buffer: stability

Theorem (Garavello-Goatin, DCDS-A 2012)

The solution $(\rho_1, \ldots, \rho_{n+m}, r)$ constructed in the previous Theorem depends on the initial condition $(\rho_{0,1}, \ldots, \rho_{0,n+m}, r_0) \in$
$\left(\prod_{i=1}^{n} BV(]-\infty, 0]; [0,1])\right) \times \left(\prod_{j=n+1}^{n+m} BV([0, +\infty[; [0,1])\right) \times [0, r_{max}]$
in a Lipschitz continuous way with respect to the strong topology of the Cartesian product $\left(\prod_{i=1}^{n} L^1(-\infty, 0)\right) \times \left(\prod_{j=n+1}^{n+m} L^1(0, \infty)\right) \times \mathbb{R}$
(with Lipschitz constant $L = 1$).

Proof: Shifts differentials.

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- A. Bressan, K.Nguyen. Conservation law models for traffic flow on a network of roads. Netw. Heter. Media 2015.
- M. Garavello, B. Piccoli. A multibuffer model for LWR road networks. In Advances in Dynamic Network Modeling in Complex Transportation Systems, Ukkusuri, Satish V.; Ozbay, Kaan (Eds.), Springer, 2013.
- M. Garavello, K. Han, B. Piccoli. Models for Vehicular Traffic on Networks. AIMS Series on Applied Mathematics, 9, American Institute of Mathematical Sciences(AIMS), Springfield, MO, 2016.
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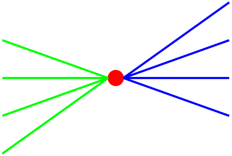
Outline of the talk

Riemann problem ARZ at J

$$\begin{cases} \partial_t \rho_k + \partial_x(\rho_k v_k) = 0 \\ \partial_t(\rho_k w_k) + \partial_x(\rho_k v_k w_k) = 0 \\ \rho_k(0,x) = \rho_{k,0}, \quad v_k(0,x) = v_{k,0} \end{cases}$$

$k = 1, \ldots, n+m$



Riemann solver:

- waves with negative speed in incoming roads
- waves with positive speed in outgoing roads
- conservation of cars: $\sum_{i=1}^{n}(\bar\rho_i \bar v_I) = \sum_{j=n+1}^{n+m}(\bar\rho_j \bar v_j)$
- drivers' preferences

$$\begin{pmatrix} \bar\rho_{n+1}\bar v_{n+1} \\ \vdots \\ \bar\rho_{n+m}\bar v_{n+m} \end{pmatrix} = A \begin{pmatrix} \bar\rho_1 \bar v_1 \\ \vdots \\ \bar\rho_n \bar v_n \end{pmatrix}$$

- $\max \sum_{i=1}^{n} \rho_i v_i$

Riemann problem ARZ at J

$$\begin{cases} \partial_t \rho_k + \partial_x(\rho_k v_k) = 0 \\ \partial_t(\rho_k w_k) + \partial_x(\rho_k v_k w_k) = 0 \\ \rho_k(0,x) = \rho_{k,0}, \quad v_k(0,x) = v_{k,0} \end{cases}$$

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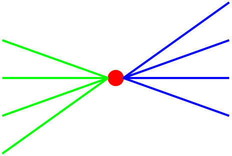

Previous rules are sufficient to isolate a unique solution in incoming roads, but not in outgoing roads.

Additional rules

- maximize the velocity v of cars in outgoing roads
- maximize the density ρ of cars in outgoing roads
- minimize the total variation of ρ along the solution of the Riemann problem in outgoing roads

Basic bibliography

ARZ:

- M. Garavello, B. Piccoli. Traffic flow on a road network using the Aw-Rascle model. Comm. Partial Differential Equations, 2006.
- M. M. Herty, S. Moutari, M. Rascle. Optimization criteria for modelling intersections of vehicular traffic flow. Netw. Heterog. Media 2006.
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- M. Garavello, F. Marcellini. The Riemann problem at a junction for a phase transition traffic model. Discrete Contin. Dyn. Syst. A 2017.
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Outline of the talk

Multi-class model on networks[6]

ρ_ℓ^c density of vehicles of class $c = 1, \dots, N_c$ on link I_ℓ
$\rho_\ell = \sum_c \rho_\ell^c$ total traffic density on link I_ℓ

$$\partial_t \rho_\ell^c + \partial_x(\rho_\ell^c v_\ell(\rho_\ell)) = 0 \qquad x \in I_\ell, t > 0,$$

[6] [Garavello-Piccoli, CMS 2005; Cristiani-Priuli, NHM 2015; Samanayarake&al, Tr. Sci. 2018]

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$$\partial_t \rho_\ell^c + \partial_x(\rho_\ell^c v_\ell(\rho_\ell)) = 0 \qquad x \in I_\ell, t > 0,$$

Summing on $c = 1, \ldots, N_c$ we get

$$\partial_t \rho_\ell + \partial_x(\rho_\ell v_\ell(\rho_\ell)) = 0 \qquad x \in I_\ell, t > 0,$$

[6] [Garavello-Piccoli, CMS 2005; Cristiani-Priuli, NHM 2015; Samanayarake&al, Tr. Sci. 2018]

Multi-class junction conditions

1. *Compose the total distribution matrix.*
   $A^c = \{a^c_{ji}\}_{i,j}$ distribution matrices for each class $c = 1, \dots, N_c$. Set

$$A := \{a_{ji}\}, \quad \text{where} \quad a_{ji} := \sum_{c=1}^{N_c} a^c_{ji} \frac{\rho^c_i}{\rho_i} \tag{1}$$

   weighted distribution matrix for the *total density* of the populations at the junction.

2. *Compute the fluxes* $(\bar{\gamma}_1, \dots, \bar{\gamma}_{n+m})$
   using the selected Riemann solver $\mathcal{RS}_J = \mathcal{RS}^A_J$ corresponding to (1).

3. *Distribute the fluxes among the various classes.*
   The incoming and outgoing fluxes for each class are given by

$$\bar{\gamma}^c_i = \frac{\rho^c_i}{\rho_i} \bar{\gamma}_i, \quad i = 1, \dots, n, \quad \bar{\gamma}^c_j = \sum_{i=1}^{n} a^c_{ji} \bar{\gamma}^c_i, \qquad j = n+1, \dots, n+m.$$

Strategy modeling on network[7]

Weighted distance from the target $\mathcal{T}^c$: u_ℓ^c viscosity solution of

$$\begin{cases} \partial_x u_\ell^c(x) + \frac{1}{g^c(x,t,\rho_\ell(x,t))} = 0 & x \in I_\ell \\ \min\limits_{\ell \in Out(J_k)} u_\ell^c(0) = u_l^c(L_l) & J_k \in \mathcal{J} \setminus \mathcal{T}^c,\ l \in Inc(J_k) \\ u_\ell^c(L_\ell) = 0\,, & \pi_\ell(L_\ell) \in \mathcal{T}^c \end{cases}$$

where g^c is the running cost ($g^c \equiv 1$ or $g^c = v$)
[Schieborn-Camilli 2013; Camilli-Festa-Schieborn 2013]

[7] [FestaGoatin, CDC2019]

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Weighted distance from the target $\mathcal{T}^c$: u_ℓ^c viscosity solution of

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where g^c is the running cost ($g^c \equiv 1$ or $g^c = v$)
[Schieborn-Camilli 2013; Camilli-Festa-Schieborn 2013]

We set

$$W_k^c := \left\{ l \in Out(J_k) : \quad u_l^c(0) = \min_{j \in Out(J_k)} u_j^c(0) \right\}$$

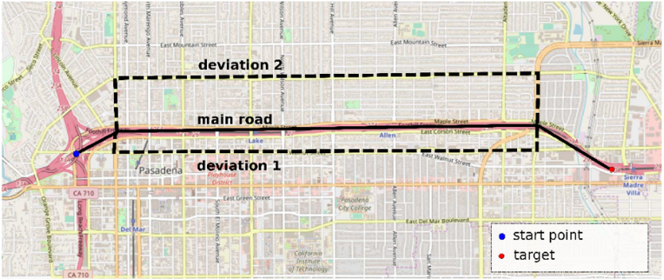
and

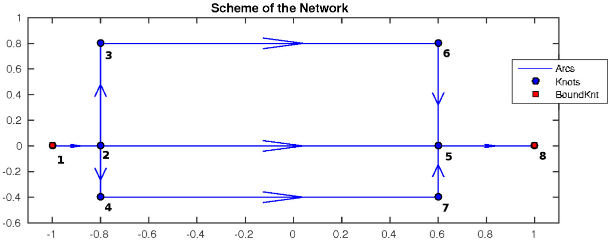
$$\alpha_{ji}^c = \begin{cases} 1/|W_k^c|, & \text{if } j \in W_k^c \\ 0, & \text{otherwise} \end{cases}$$

[7] [FestaGoatin, CDC2019]

Outline of the talk

Example 1[8]: Pasadena





[8] [Thai-LaurentBrouty-Bayen, IEEE ITS 2016]

Example 1

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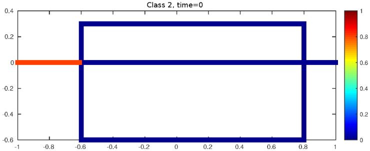

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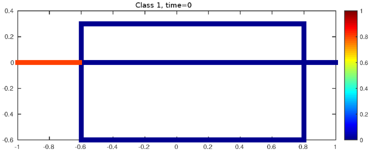

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Example 1

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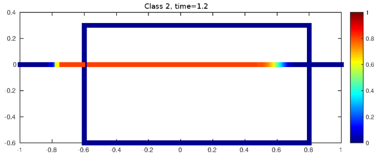

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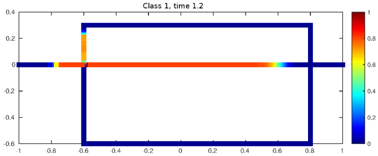

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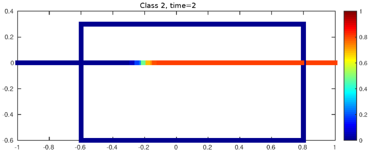

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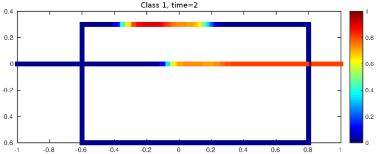

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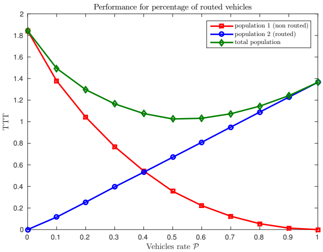
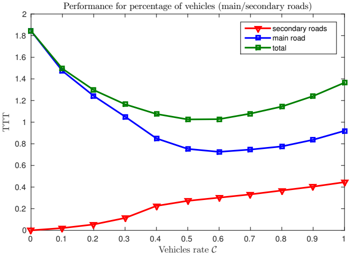
Example 1

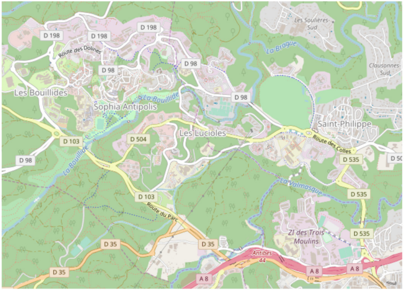


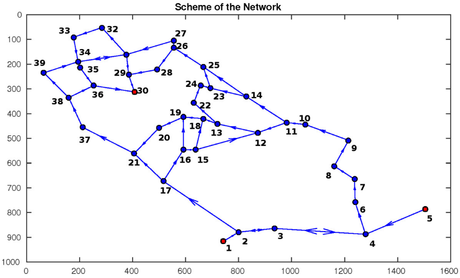
Total Travel Time in the whole network for each of the two populations and for the whole population depending on the penetration rate of routed vehicles $\mathcal{P}$



Total Travel Time in the main road and in the two detours to reach destination depending on the penetration rate of routed vehicles $\mathcal{P}$

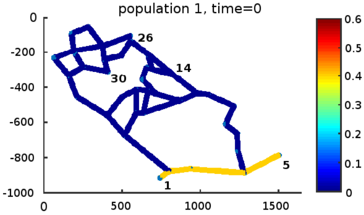
Example 2: Sophia Antipolis





Example 2

P=0.5



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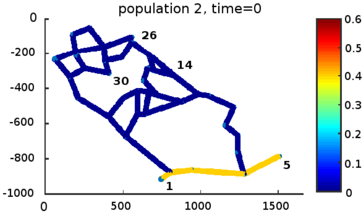

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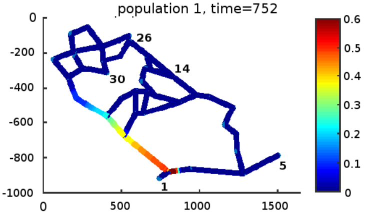
Example 2

P=0.5



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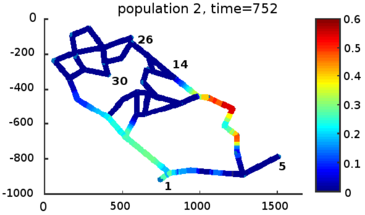

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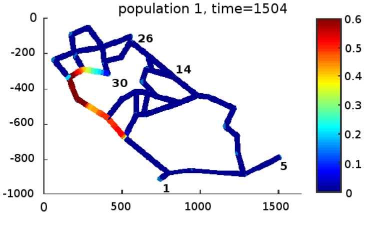

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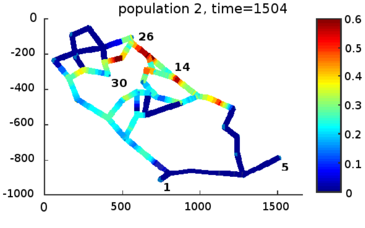

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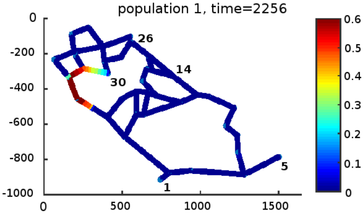

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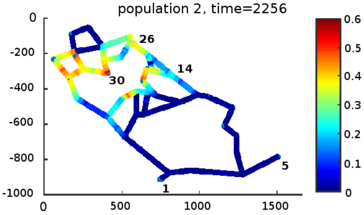

informed

Conclusion

Multi-population model accounting for routing choices:

- Can be applied to any Riemann Solver at junction
- Solves eikonal equations on networks
- Reproduces expected behaviour
- Can be extended to route choice based on traffic forecast
- Convergence?

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Multi-population model accounting for routing choices:

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Thank you!