Graphon games: A statistical framework for network games and interventions

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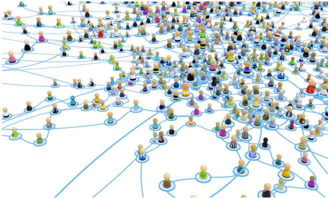
Motivation



Social interactions

- Adoption of innovations, behaviors
- Opinion formation
- Social learning

Economic interactions

- Public good provision
- Competition among firms
- Financial trades

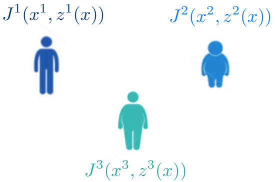
In many social and economic settings, decisions of individuals are affected more by the actions of their friends, colleagues, peers and competitors.

Network game model

Consider a network game defined by:

- N agents
- interacting over a network $G \in \mathbb{R}^{N \times N}$

$$\begin{cases} G_{ij} \geq 0 & \text{influence of } j \text{ on } i \\ G_{ii} = 0 & \text{no self loops} \end{cases}$$



Each agent i aims at minimizing its cost function

- strategy: $x^i \in \mathbb{R}^n$
- feasible set: $\mathcal{X}^i \subset \mathbb{R}^n$
- cost: $J^i(x^i, z^i(x)) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
- aggregate: $z^i(x) := \frac{1}{N} \sum_{j=1}^{N} G_{ij} x^j$

Standing assumption

- $\mathcal{X}^i \subset \mathbb{R}^n$ compact and convex;
- $J^i(x^i, z^i(x))$ strongly convex in x^i, for all $x^{-i} \in \mathcal{X}^{-i}$;
- $J^i(x^i, z^i) \in \mathcal{C}^2$ in $[x^i; z^i]$.

Linear quadratic network games

- Each agent chooses **an action** $x^i \geq 0$ ...

  $\rightarrow$ how much effort exerted on an activity
  (e.g. education, smoking, public goods)

- Agent i **cost function**:

$$J^i(x^i, z^i(x)) = \underbrace{\frac{1}{2}(x^i)^2 - a^i x^i}_{\text{cost isolation}} - \underbrace{K \cdot z^i(x)\ x^i}_{\text{network effects}}$$

  - aggregate: $z^i(x) = \frac{1}{N} \sum_{j \neq i} G_{ij} x^j$
  - K determines how much neighbor actions affect agent's payoff.
    ($K > 0$ strategic complements; $K < 0$ strategic substitutes)

A set of strategies $\{\bar{x}^i\}_{i=1}^N$ is a **Nash equilibrium** if for each player i,

$$J^i\left(\bar{x}^i, z^i(\bar{x})\right) \leq J^i\left(x^i, z^i(\bar{x})\right), \quad \text{for all } x^i \in \mathcal{X}^i.$$

Literature and main question

What is the impact of network structure on equilibrium outcome?

- How does individual network position determine individual play?

  Ballester et al. (2006); Bramoullé and Kranton (2007); Bramoullé et al. (2014); Belhaj et al. (2014); Jackson and Zenou (2014); Acemoglu et al. (2015); Allouch (2015); Melo (2017); Parise and Ozdaglar (2018)

- How does a central planner target interventions?

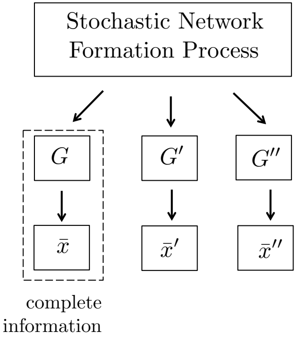
  - Ballester et al. (2006): key-player removal in crime applications
  - Candogan et al. (2012): optimal pricing for monopolist
  - Galeotti et al. (2017): budget allocation in network games

  $\hookrightarrow$ require exact network information

Applications where network is large, changing over time or multiple networks

Can we regulate strategic behavior by using only statistical information about network interactions?

A statistical framework for network games



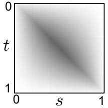
Sampled network game = game over a sampled network

- Related work on distribution of centrality measures: Dasaratha (2017), Avella-Medina, Parise, Schaub, and Segarra (2018)

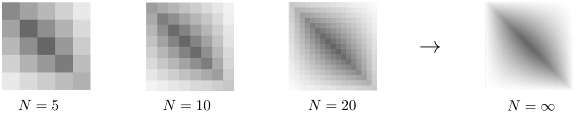
Graphons as stochastic network formation processes

Graphons -[Lovász, 2012]

A symmetric measurable function $W : [0,1]^2 \to [0,1]$



1) Limit of graph: $W(s,t) =$ interaction $s, t \in [0,1]$



Graphons as stochastic network formation processes

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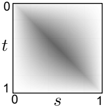
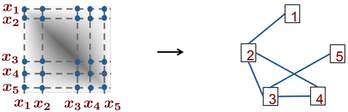
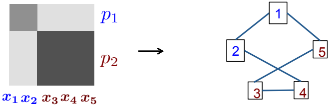


2) Random graph model:



Generalize **parametric models** such as Erdos-Renyi, Stochastic Block model



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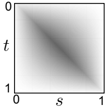
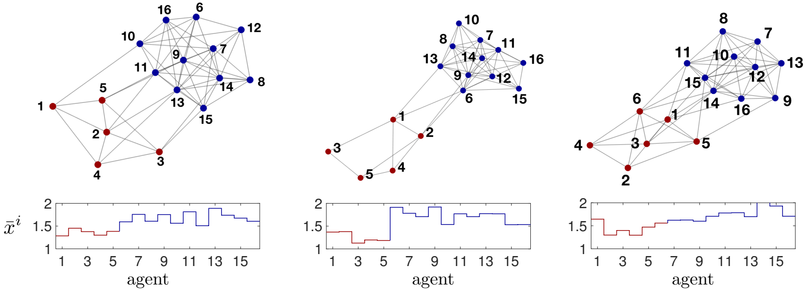


- Theory:
  [Lovász, Szegedy, 2006], [Lovász, 2012], [Borgs et al., 2008]
- Applications:
  - community detection [Eldridge et al., 2016],
  - crowd-sourcing [Lee and Shah, 2017],
  - signal processing [Morency and Leus, 2017],
  - optimal control of dynamical systems [Gao and Caines, 2017]
  - graphon mean field games: [Caines and Huang, 2018]
  - ...
- Key idea of this work:

combine network game theory with graphon theory

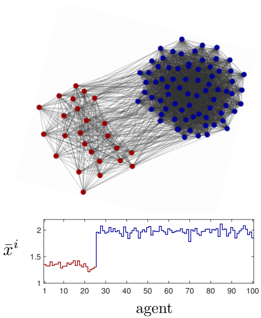
Illustration for a SBM

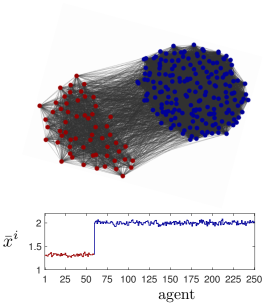


Questions:

1. How close are Nash equilibria in different sampled network games?

Illustration for a SBM

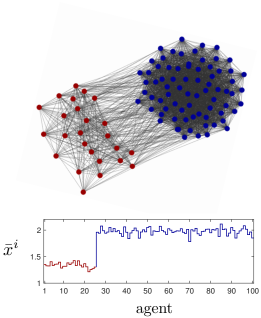


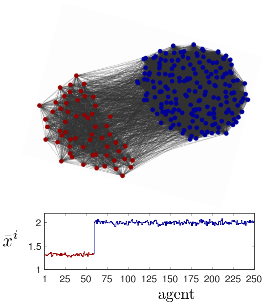


Questions:

1. How close are Nash equilibria in different sampled network games?
2. Will the Nash equilibria converge to a deterministic profile for N large?

Illustration for a SBM

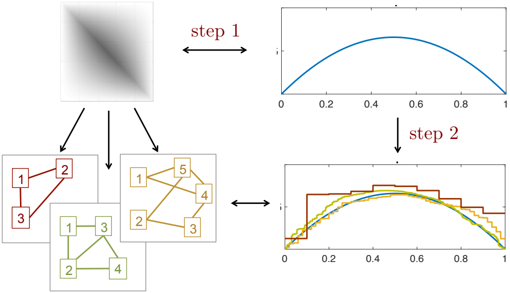




Questions:

1. How close are Nash equilibria in different sampled network games?
2. Will the Nash equilibria converge to a deterministic profile for N large?
3. Can we exploit this property to design robust interventions for sampled network games?

Talk outline



Step 1 Define graphon games for infinite populations

- define equilibrium
- existence, uniqueness and sensitivity

Step 2 Relate infinite graphon games to sampled network games

- reformulate a network games as a step-function graphon game
- relate equilibria of graphon games & sampled network games (bound the distance in terms on the population size)

Step 3 Design interventions for sampled network games based on graphon model

Step 4 Incomplete information in sampled network games

Step 1:
Infinite population

Network versus graphon games

	Network games	**Graphon games**
- Agents:	$i \in \{1, \dots, N\}$	$s \in [0, 1]$
- Interactions:	$G \in \mathbb{R}^{N \times N}$	$W : [0, 1]^2 \to [0, 1]$
- Strategy:	$x^i \in \mathcal{X}^i$	$x(s) \in \mathcal{X}(s)$
- Cost function:	$J(x^i, z^i)$	$J(x(s), z(s \mid x))$
- Aggregate:	$z^i(x) := \frac{1}{N} \sum_{j=1}^{N} G_{ij} x^j$	$z(s \mid x) := \int_0^1 W(s, t) x(t) dt$

Remarks:

- Agents are a continuum in $[0, 1]$ (non-atomic)
- $W(s, t)$ represents the interaction among the non-atomic agents s and t
- The agents cost function is the same in network and in graphon games
- A strategy profile is a function $x : [0, 1] \to \mathcal{X}$

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Definition: Nash equilibrium

A function $\bar{x}(s) \in \mathcal{X}(s)$ is a Nash equilibrium if for all $s \in [0, 1]$

$$J(\bar{x}(s), z(s \mid \bar{x})) \leq J(\tilde{x}, z(s \mid \bar{x})) \text{ for all } \tilde{x} \in \mathcal{X}(s)$$

Similarity with Wardrop equilibrium for non-atomic congestion games [Wardrop, 1900], [Smith, 1979]

Network versus graphon games

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- Agents:	$i \in \{1, \dots, N\}$	$s \in [0, 1]$
- Interactions:	$G \in \mathbb{R}^{N \times N}$	$W : [0, 1]^2 \to [0, 1]$
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Does a Nash equilibrium exist? Is it unique?

The graphon operator

Adjacency matrix

$G \in \mathbb{R}^{N\times N}$

$v \mapsto Gv$

$Gv = \lambda v$

Graphon operator

$\mathbb{W} : L^2([0,1]) \mapsto L^2([0,1])$

$f \mapsto (\mathbb{W}f)(s) = \int_0^1 W(s,t)f(t)\mathrm{d}t.$

$(\mathbb{W}f)(s) = \lambda f(s)$

Properties of $\mathbb{W}$ - [Lovász, 2012]

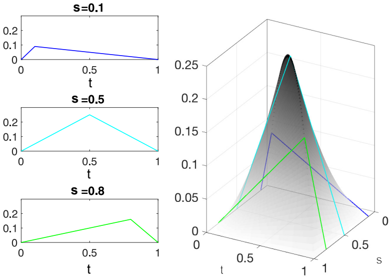
1. $\mathbb{W}$ is a linear, continuous, bounded operator;
2. all the eigenvalues of $\mathbb{W}$ are real;
3. $|||\mathbb{W}||| := \sup_{f\in L^2([0,1]), \|f\|_L=1} \|\mathbb{W}f\|_L = \lambda_{\max}(\mathbb{W})$.

An example

Consider an infinite population of agents which are spatially located along a line (e.g. a street).

- $s \in [0, 1]$ = position along line
- influence between agents is a decreasing function of the spatial distance
- central agents are affected more

minmax graphon



$$W(s, t) = \min(s, t)(1 - \max(s, t))$$

Spectral properties:

$$\lambda_h := \frac{1}{\pi^2 h^2}, \qquad \psi_h(s) := \sqrt{2} \sin(h\pi s) \quad \forall h \in \{1, 2, \ldots \infty\}$$

W has an infinite, but countable, number of nonzero eigenvalues, with an accumulation point at zero. Moreover, $\lambda_{\max}(\mathbb{W}) = \frac{1}{\pi^2}$.

Reformulation as fixed point of the best response mapping

Nash equilibrium

$$\bar{x}(s) = \arg\min_{\tilde{x}\in\mathcal{X}(s)} J(\tilde{x}, z(s \mid \bar{x}))$$

- consider a fixed strategy profile $x(s)$
- the corresponding local aggregate function is

$$z_x(s) := z(s \mid x) = \int_0^1 W(s,t)x(t)dt = (\mathbb{W}x)(s)$$

- Define the best response operator $\mathbb{B}$

$$(\mathbb{B}z)(s) := \arg\min_{\tilde{x}\in\mathtt{X}(s)} J(\tilde{x}, z(s)),$$

- the best response mapping is

$$x \mapsto \mathbb{B}z_x = \mathbb{B}\mathbb{W}x$$

Lemma - Equivalent characterization

$\bar{x}$ is a Nash equilibrium iff it is a fixed point of the game operator $\mathbb{B}\mathbb{W}$, i.e.

$$\bar{x} = \mathbb{B}\mathbb{W}\bar{x}.$$

Existence and uniqueness

Assumption on cost and strategy sets

- $J(x,z)$ is $\mathcal{C}^1$ and strongly convex in x uniformly in z (constant μ_J)
  $\nabla_x J(x,z)$ is Lipschitz in z uniformly in x (constant ℓ_J)
- $\mathtt{X}(s)$ convex and closed $\forall s \in [0,1]$, $\exists\, \mathcal{X}$ compact s.t. $\mathtt{X}(s) \subseteq \mathcal{X}, \forall s$.

Proof idea: prove that $\mathbb{B}\mathbb{W}$ is a contraction

1. prove that $\mathbb{B}$ is Lipschitz

$$\|\mathbb{B}z_1 - \mathbb{B}z_2\|_{L^2} \leq \frac{\ell_J}{\mu_J}\|z_1 - z_2\|_{L^2}$$

2. combining with $\mathbb{W}$ we get

$$\begin{aligned}\|\mathbb{B}\mathbb{W}x_1 - \mathbb{B}\mathbb{W}x_2\|_{L^2} &\leq \frac{\ell_J}{\mu_J}\|\mathbb{W}x_1 - \mathbb{W}x_2\|_{L^2} = \frac{\ell_J}{\mu_J}\|\mathbb{W}(x_1 - x_2)\|_{L^2} \\ &\leq \frac{\ell_J}{\mu_J}|\!|\!|\mathbb{W}|\!|\!|\,\|x_1 - x_2\|_{L^2} = \frac{\ell_J}{\mu_J}\lambda_{\max}(\mathbb{W})\|x_1 - x_2\|_{L^2}\end{aligned}$$

3. apply Banach fixed point theorem

Theorem

$$\frac{\ell_J}{\mu_J}\lambda_{\max}(\mathbb{W}) < 1 \quad \Rightarrow \quad \text{existence and uniqueness}$$

Linear quadratic graphon games

$$J(x^i, z^i) = \frac{1}{2}(x^i)^2 - x^i[Kz^i + a]$$

- a Nash equilibrium exists and is unique if

$$\frac{\ell_J}{\mu_J}\lambda_{\max}(\mathbb{W}) < 1 \quad \Leftrightarrow \quad K < \frac{1}{\lambda_{\max}(\mathbb{W})}$$

$\rightarrow$ compare with: [Ballester et al., 2006], [Jackson and Zenou, 2014]

- for game of strategic complements ($K > 0$) equilibrium is proportional to Bonacich centrality

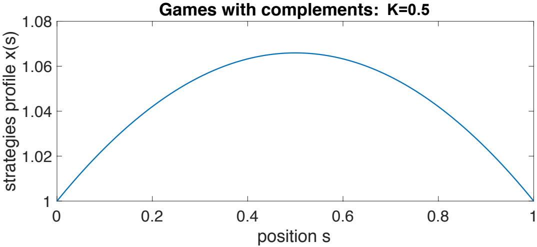
$$\bar{x}(s) = a((\mathbb{I} - K\mathbb{W})^{-1}1_{[0,1]})(s)$$

$\rightarrow$ centrality measures for graphons:
[Avella-Medina, Parise, Schaub, Segarra. 2017]

Example - cont'd

$$J(x^i, z^i) = \frac{1}{2}(x^i)^2 - x^i[Kz^i + a]$$

- Use minmax graphon and recall $\lambda_{max}(\mathbb{W}) = \frac{1}{\pi^2}$
- Set $K = 0.5$ (for uniqueness).



Comparative statics

How does the equilibrium change if the graphon changes from $\mathbb{W}$ to $\tilde{\mathbb{W}}$?

Theorem

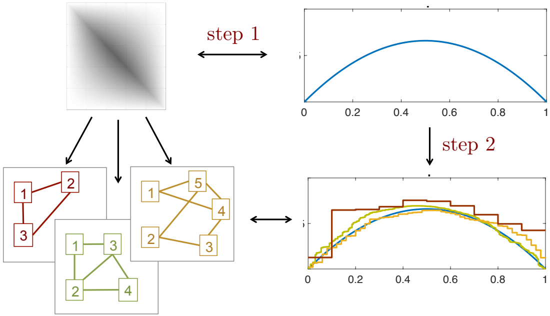
Let $x_{\max} := \max_{x \in \mathcal{X}} \|x\|$. Then under the previous assumptions

$$\|\bar{x} - \tilde{x}\|_{L^2} \leq \frac{\ell_J/\mu_J x_{\max}}{1 - \ell_J/\mu_J \lambda_{\max}(\mathbb{W})} \left\|\left\| \mathbb{W} - \tilde{\mathbb{W}} \right\|\right\|$$

Proof idea:

$$\begin{aligned}
\|\bar{x} - \tilde{x}\|_{L^2} = \|\mathbb{B}\mathbb{W}\bar{x} - \mathbb{B}\tilde{\mathbb{W}}\tilde{x}\|_{L^2} &\leq \frac{\ell_J}{\mu_J}\|\mathbb{W}\bar{x} - \tilde{\mathbb{W}}\tilde{x}\|_{L^2} \\
&\leq \frac{\ell_J}{\mu_J}\|\mathbb{W}\bar{x} - \mathbb{W}\tilde{x}\|_{L^2} + \frac{\ell_J}{\mu_J}\|\mathbb{W}\tilde{x} - \tilde{\mathbb{W}}\tilde{x}\|_{L^2} \\
&\leq \frac{\ell_J}{\mu_J}|||\mathbb{W}|||\,\|\bar{x} - \tilde{x}\|_{L^2} + \frac{\ell_J}{\mu_J}\left\|\left\|\mathbb{W} - \tilde{\mathbb{W}}\right\|\right\| \|\tilde{x}\|_{L^2}
\end{aligned}$$

Step 2:
finite population



Step 2: Relation sample network and graphon game

Theorem: Nash equilibrium distance

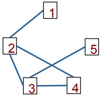
Suppose that W is Lipschitz continuous and fix any tolerance $\delta \ll 1$.

With probability at least $1 - 2\delta$

$$\|\bar{x}^{[N]} - \bar{x}\|_{L^2} \leq K\sqrt{\frac{\log(N/\delta)}{N}}$$

Proof idea:

- map any finite network game to a graphon game with



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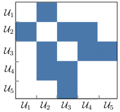


piece-wise constant graphon $W^{[N]}$

- relate equilibria distance to graphon operator distance:

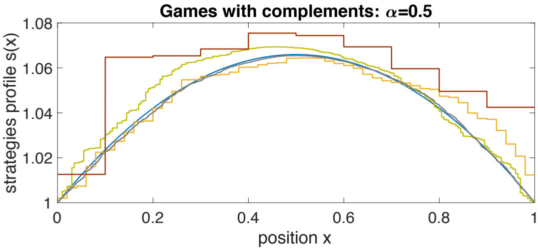
$$\|\bar{x}^{[N]} - \bar{x}\|_{L^2} \leq \tilde{K}_1 \left|\left|\left| \mathbb{W}^{[N]} - \mathbb{W} \right|\right|\right|$$

- bound the graphon operator distance (improvement on [Lovász, 2012]):

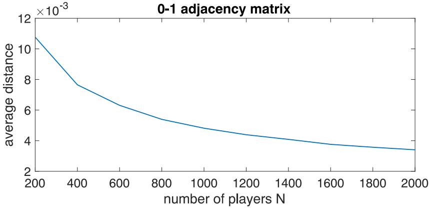
$$\left|\left|\left| \mathbb{W}^{[N]} - \mathbb{W} \right|\right|\right| \leq \tilde{K}_2 \sqrt{\frac{\log(N/\delta)}{N}}$$

Example - cont'd

Use minmax graphon and recall $\lambda_{max}(\mathbb{W}) = \frac{1}{\pi^2}$. Set $\alpha = 0.5$ (for uniqueness).



- Plot the equilibrium in sampled network games for $N = 10, 50, 200, 2000$
- Plot expected distance over 100 realizations



Step 3:
Interventions

Welfare maximization in LQ network games

$$J(x^i, z^i) = \frac{1}{2}(x^i)^2 - x^i[Kz^i + a^i], \qquad K > 0$$

This could model for example peer pressure in education: Calvó-Armengol, Patacchini, Zenou (2009)

- x^i = student effort
- K = level of peer pressure
- a^i = effort in isolation

Welfare maximization in LQ network games

$$J(x^i, z^i \mid \beta^i) = \frac{1}{2}(x^i)^2 - x^i[Kz^i + a^i + \beta^i]$$

Per-capita welfare maximization problem - Galeotti et al., (2017):

$$\max_{\beta \in \mathbb{R}^N} \quad T_\beta^{[N]} := -\frac{1}{N}\sum_{i=1}^N J(\bar{x}^i, \bar{z}^i \mid \beta^i)$$

$$\text{s.t.} \quad \sum_{i=1}^N (\beta^i)^2 \leq C^{[N]},$$

Network heuristic

$$\beta_{\text{nh}}^{[N]} := \sqrt{C^{[N]}} v_1^{[N]}$$

where $v_1^{[N]}$ is the dominant eigenvector of $G^{[N]}$

Graphon heuristic

$$[\beta_{\text{gh}}^{[N]}]_i := \kappa^{[N]} \cdot \psi_1(s_i),$$

where ψ_1 is the dominant eigenfunction of W

Performance of the graphon heuristic

Theorem

If further $\lambda_1(\mathbb{W}) > \lambda_2(\mathbb{W})$ and $C^{[N]} = \mathcal{O}(N)$, with probability $1 - 2\delta$

$$|T_{nh}^{[N]} - T_{gh}^{[N]}| = \mathcal{O}\left(\sqrt{\frac{\log(N/\delta)}{N}}\right).$$

Proof idea:

- i) $T^{[N]} = \frac{1}{2N}\|\bar{x}^{[N]}\|^2$ and ii) $\bar{x}^{[N]} = [I - \alpha \frac{G^{[N]}}{N}]^{-1}(a + \beta)$

$$\begin{aligned}
|T_{nh}^{[N]} - T_{gh}^{[N]}| &\le \frac{\sqrt{C^{[N]}} + a\sqrt{N}}{N} \frac{1}{(1-\eta K \lambda_1(\mathbb{W}))^2} \|\beta_{\text{nh}}^{[N]} - \beta_{\text{gh}}^{[N]}\| \\
&\approx \frac{\sqrt{C^{[N]}} + a\sqrt{N}}{N} \frac{\sqrt{C^{[N]}}}{(1-\eta K \lambda_1(\mathbb{W}))^2} \|\varphi_1^{[N]} - \varphi_1\|_{L^2}
\end{aligned}$$

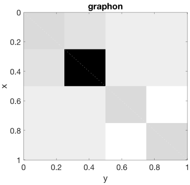
- By Davis-Kahan theorem

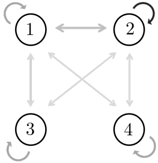
$$\| \underbrace{\varphi_1^{[N]}}_{\text{rel. to } \mathbb{W}^{[N]}} - \underbrace{\varphi_1}_{\text{rel. to } \mathbb{W}} \|_{L^2} \le \frac{2\sqrt{2}\,\left|\!\left|\!\left| \mathbb{W}^{[N]} - \mathbb{W} \right|\!\right|\!\right|}{\lambda_1(\mathbb{W}) - \lambda_2(\mathbb{W})} = \mathcal{O}\left(\sqrt{\frac{\log(N/\delta)}{N}}\right)$$

The community model

SBM model

- generalize to K communities
- each agent belongs to community k with probability w_k
- agents in community l, k connect with probability $q_{l,k}$



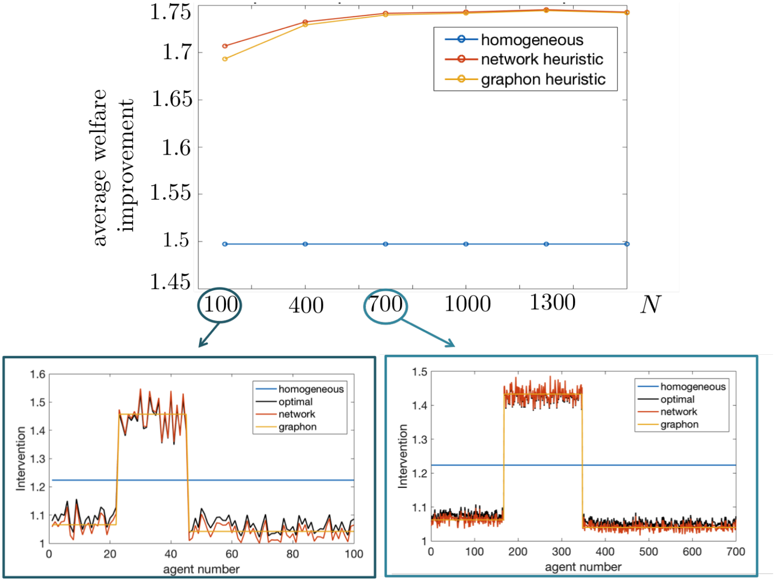


(e.g. Currarini et al. (2009) community= students of different race)

How to compute the dominant eigenfunction?

- Let $D := \text{diag}([w_k]) \in \mathbb{R}^{4\times 4}$
- Let $Q := [q_{l,k}] \in \mathbb{R}^{4\times 4}$
- Let v_1 dominant eigenvector of $QD \in \mathbb{R}^{4\times 4}$
- Then $\psi_1(s)$ is piece-wise constant with values given by v_1

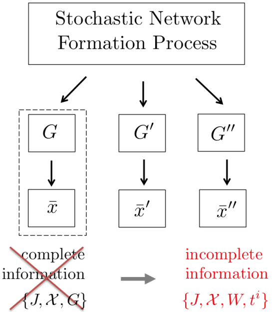
The community model - cont'd



The graphon heuristic is much simpler to implement

Step 4:
Incomplete information

Incomplete information in sampled network games



An incomplete information sampled network game $\mathcal{G}^{in}(\mathbb{X}, J, W)$ is a game

- with a random number N of agents
- with types $\{t^i\}_{i=1}^N$ sampled i.i.d from $\mathcal{U}[0,1]$
- interacting according to a network $G^{[N]}$ sampled from the graphon W
- each agent i has information about: W, t^i, $\mathcal{X}$ and J
- while is uninformed about $G^{[N]}$ and the other agents types t^{-i}

Symmetric Bayesian Nash equilibrium

- Suppose agent of type s play $b(s)$ (symmetric case)
- The expected cost of an agent of type $t^i = s$ playing $x(s) \in \mathbb{X}(s)$ is

$$J_{\exp}(x(s) \mid b) = \mathbb{E}_{N,t^{-i},\text{links}}\left[J\left(x(s), \frac{1}{N-1}\sum_{j\neq i}[G^{[N]}]_{ij}b(t^j)\right)\right]$$

Symmetric Bayesian Nash equilibrium

$b(s) \in \mathbb{X}(s)$ is a symmetric ε-Bayesian Nash equilibrium if for all $s \in [0,1]$

$$J_{\exp}(b(s) \mid b) \leq J_{\exp}(\tilde{x} \mid b) + \varepsilon \text{ for all } \tilde{x} \in \mathbb{X}(s).$$

Symmetric Bayesian Nash eq. are strictly related to graphon Nash eq.

Linear quadratic games

Theorem

$\bar{x}$ is a Nash equilibrium of $\mathcal{G}(\mathtt{X}, J, W)$ iff it is a symmetric Bayesian Nash equilibrium of $\mathcal{G}^{in}(\mathtt{X}, J, W)$.

Proof idea:

- $\bar{x}$ *Graphon equilibrium iff* $\boldsymbol{J(\bar{x}(s), \bar{z}(s)) \leq J(\tilde{x}, \bar{z}(s))}, \forall \tilde{x}, s$

$$\bar{z}(s) = \int_0^1 W(s,t)\bar{x}(t)dt$$

- Fix $b = \bar{x}$, by linearity in aggregate: $J_{\text{exp}}(x(s) \mid \bar{x}) = J(x(s), z_{\text{exp}}(s))$ where

$$z_{\text{exp}}(s) := \mathbb{E}_{N,t^{-i},\text{links}} \left[\frac{1}{N-1} \sum_j [G^{[N]}]_{ij} \bar{x}(t^j) \right]$$

$\bar{x}$ *Bayesian equilibrium iff* $\boldsymbol{J(\bar{x}(s), z_{\text{exp}}(s)) \leq J(\tilde{x}, z_{\text{exp}}(s))}, \forall \tilde{x}, s.$

- Conclusion follows from $z_{\text{exp}}(s) = \bar{z}(s)$

$$z_{\text{exp}}(s) = \mathbb{E}_{\text{N}} \mathbb{E}_{t^{-i}|\text{N}} \mathbb{E}_{\text{links}|t^{-i},N} \left[\frac{1}{N-1} \sum_{j \neq i} [G^{[N]}]_{ij} \bar{x}(t^j) \right]$$

$$= \mathbb{E}_{\text{N}} \mathbb{E}_{t^{-i}|\text{N}} \left[\frac{1}{N-1} \sum_{j \neq i} W(s,t^j)\bar{x}(t^j) \right] = \mathbb{E}_{\text{N}} \frac{1}{N-1} \sum_{j \neq i} \mathbb{E}_{t^j} \left[W(s,t^j)\bar{x}(t^j) \right]$$

$$= \mathbb{E}_{\text{N}} \frac{1}{N-1} \sum_{j \neq i} \int_0^1 W(s,t)\bar{x}(t)dt = \mathbb{E}_{\text{N}} \frac{1}{N-1} \sum_{j \neq i} \bar{z}(s) = \mathbb{E}_{\text{N}} \bar{z}(s) = \bar{z}(s)$$

Generalization to Lipschitz cost

Theorem

Further suppose that

- $J(x,z)$ is Lipschitz continuous in z uniformly over x
- agents know that $N \geq N_{\text{min}}$

The Nash equilibrium of $\mathcal{G}(\mathtt{X}, J, W)$ is a symmetric ε-Bayesian Nash equilibrium with

$$\varepsilon = \mathcal{O}\left(\sqrt{\frac{\log(N_{\text{min}})}{N_{\text{min}}}}\right).$$

Proof idea:

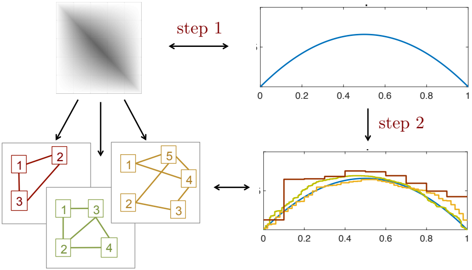
- For general cost

$$J_{\text{exp}}(x(s) \mid \bar{x}) = \mathbb{E}_{N,t^{-i},\text{links}}\left[J\left(x(s), \tfrac{1}{N-1}\sum_j [G^{[N]}]_{ij}\bar{x}(t^j)\right)\right] \neq J\left(x(s), z_{\text{exp}}(s)\right)$$
$$= \mathbb{E}_{\zeta_{\bar{x}}(s)}\left[J\left(x(s), \zeta_{\bar{x}}(s)\right)\right]$$

- Prove that $\zeta_{\bar{x}}(s)$ concentrates around $z_{\text{exp}}(s)$ for N large
- Use Lipschitz condition to show that

$$J_{\text{exp}}(x(s) \mid \bar{x}) \approx J\left(x(s), z_{\text{exp}}(s)\right) = J\left(x(s), \bar{z}(s)\right)$$

Conclusion



Summary

- Define graphon games and study equilibrium properties
- Graphon equilibrium is a good approximation for sampled network games
- Shown how to design robust interventions using graphon model
- Foundation for graphon equilibrium in incomplete information games