Diffusion in large networks: is polarization possible?

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Introduction

- *Diffusion in networks*: a widely studied topic (see surveys in Jackson (2008), Bramoullé et al. (2016), Acemoglu & Ozdaglar (2011), etc.) with many applications (adoption of new technology, opinion formation, fashion, contagion and disease infection, etc.)
- Our point of departure is the work of Morris (2000) who characterizes the contagion threshold.
- G–R (2013) investigate a model of *influence based on aggregation functions,* where each agent modifies his opinion by aggregating the current opinions of all agents, and Foerster–G–R (2013) model *anonymous influence by OWA operators.*
- We apply the model of G–R (2013) to model diffusion in countably infinite networks, with arbitrary structure.
- Our aim is to study the evolution in the long run of the diffusion process, in particular to answer the following questions:

  How does the diffusion evolve from a finite set of active agents?
  Is polarization possible? Can we have cycling phenomena?

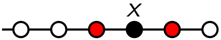
Outline

Countable networks

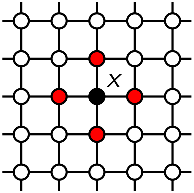
- Let $\mathcal{X}$ be the set of agents, where $\mathcal{X}$ is countably infinite (e.g., $\mathbb{Z}^2$)
- We define as in (Morris, 2000) the *neighborhood relation* $\sim$ *on* $\mathcal{X}$: $x \sim y$ (x is neighbor with y) is a binary relation which is:
  - irreflexive
  - symmetric
  - *bounded*: each x has at most M neighbors, where M is a fixed constant
  - *connected*: for every $x, y \in \mathcal{X}$, there exists a finite path connecting x to y, i.e., there exists $x_1, \ldots, x_k \in \mathcal{X}$ such that $x_1 = x, x_k = y$ and $x_i \sim x_{i+1}$ for each $i = 1, \ldots, k-1$.
- $(\mathcal{X}, \sim)$ is called a *(countable) network*.
- The *neighborhood of x* is $\Gamma(x) = \{y \,:\, x \sim y\}$. Note that $\Gamma(x)$ is finite and $x \notin \Gamma(x)$.

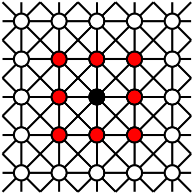
Examples

- $\mathcal{X} = \mathbb{Z}$, $x \sim y$ if $|x - y| \leq 1$



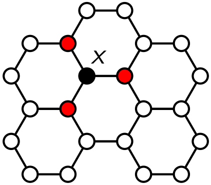
- $\mathcal{X} = \mathbb{Z}^2$, and $x \sim y$ if $d(x, y) \leq \theta$ (d is, e.g., the Euclidean distance).
  - $\theta = 1$: 4 neighbors (north, south, east, west)
  - $\theta = \sqrt{2}$: 8 neighbors (4 on the diagonals in addition)





Examples

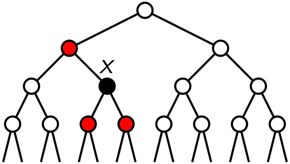
- the hexagonal pavement: each x has 3 neighbors



- $\mathcal{X} = \mathbb{Z}^d$, $d \geq 1$, and the neighborhood is defined by, e.g., the Euclidean distance

Examples

- Hierarchy: each player has m subordinates, and one superior (except the root).



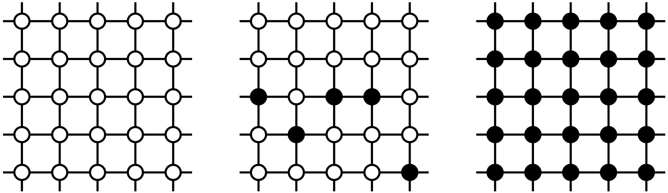
The society

- Each agent can have two statuses 0 and 1 (opinion on a given subject, adoption of a new technology, infection by some disease, etc.).
- If the status is 1, the agent is *active*, otherwise the agent is *inactive*.
- Let $\Omega = \{0,1\}^{\mathcal{X}}$ be the set of all possible *configurations* of activity:

$$\text{for } \omega \in \Omega,\ \omega(x) = \begin{cases} 1 \text{ if agent } x \text{ is active,} \\ 0 \text{ if inactive} \end{cases}$$

- Any configuration ω can be seen as the set X of its active agents by

$$1_X \equiv \omega$$



$\mathbf{0}$ or $\emptyset$ $\qquad$ ω or X $\qquad$ $\mathbf{1}$ or $\mathcal{X}$

The diffusion process: aggregation functions

- *An aggregation function with n entries* is a mapping $A : [0,1]^n \to [0,1]$ which is nondecreasing in each variable, $A(1,\dots,1) = 1$, $A(0,\dots,0) = 0$.
- It is *symmetric* or *anonymous* if $A(z_1,\dots,z_n) = A(z_{\sigma(1)},\dots,z_{\sigma(n)})$ for every permutation σ on $[n]$.

We distinguish 3 cases:

1. A is strict: $A(z) = 0$ iff z is the 0 vector and $A(z) = 1$ iff z is the 1 vector;
2. A is 0-1-valued (Boolean aggregation function)
3. A is nonstrict and nonBoolean, i.e., there exist $z \in [0,1]^n$ s.t. $0 < A(z) < 1$, and there exist $z' \neq 0$ s.t. $A(z') = 0$, or $z'' \neq 1$ s.t. $A(z'') = 1$.

The diffusion process: Informal definition

- The probability for an agent x to be active at time $t+1$ given the configuration ω at time t is given by

$$P(x \mid \omega) = A(\omega|_{\Gamma(x)})$$

- We assume that the probabilities conditionally on ω are independent.
- *trajectory:* sequence $\omega(0), \omega(1), \ldots, \omega(t), \ldots$ s.t. $|\omega(t)| < \infty \ \forall t$, with a positive probability of transition from one to the next.

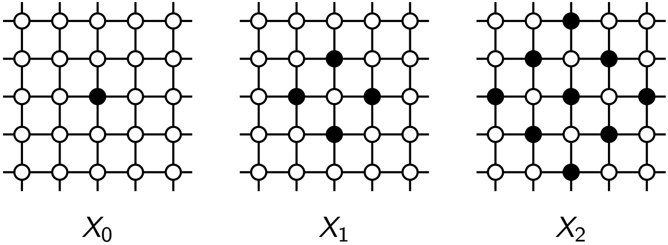


Figure: Example of a trajectory with $\mathbb{Z}^2$ and the NESW neighborhood

The diffusion process: Formal definition (1/2)

Markov process on Ω, but Ω is uncountable.

- It requires working on σ-fields.
- Given X, Y disjoint subsets of $\mathcal{X}$, the *cylinder* $(X, Y)^+$ is defined by

$$\{\omega \in \Omega, \omega|_X = 1 \text{ and } \omega|_Y = 0\}.$$

- Let $\mathcal{T}$ be the σ-field generated by the finite cylinders.
- We define a Markov Kernel on Ω, i.e., a mapping K from $\Omega \times \mathcal{T}$ to $[0, 1]$ such that
  - For every $\omega \in \Omega$, $K(\omega, \cdot)$ is a probability measure on $\mathcal{T}$,
  - For every $\mathcal{A} \in \mathcal{T}$, $K(\cdot, \mathcal{A})$ is measurable.
- $K(\omega, \mathcal{A})$ can be interpreted as the probability that from configuration ω the process jumps at next time step into a configuration belonging to $\mathcal{A}$.

The diffusion process: Formal definition (2/2)

- Fix $\omega \in \Omega$.
- For any finite $Y \subset \mathcal{X}$, consider $\{0,1\}^Y$ the set of partial configurations on Y.
- Let $\mu_{Y,\omega}$ be the probability distribution on partial configurations h defined by

$$\mu_{Y,\omega}(\{h\}) = \prod_{y \in Y} \big(P(y \mid \omega) h(y) + (1 - P(y \mid \omega))(1 - h(y)) \big)$$

  where $P(y \mid \omega) = A(\omega|_{\Gamma(y)})$.
- $\mu_{Y,\omega}$ only depends on ω restricted to Y and its neighbors.

$(\mu_{Y,\omega})_{Y \subset \mathcal{X}}$ satisfies the assumption of Kolmogorov extension theorem hence can be extended into $K(\omega, .)$ over $(\Omega, \mathcal{T})$. Moreover, $K(\cdot, \mathcal{A})$ is measurable.

Outline

1. The model
2. **General results**
3. Strict aggregation functions
4. Boolean anonymous aggregation functions
5. Conclusion

- Given $X \subseteq \mathcal{X}$, we define its closure $\overline{X}$ and interior $\mathring{X}$ by

$$\overline{X} = \{x \in \mathcal{X} : \Gamma(x) \cap X \neq \varnothing\} \text{ and } \mathring{X} = \{x \in \mathcal{X} : \Gamma(x) \subseteq X\}$$

- We have $\mathring{X} \subseteq \overline{X}$, but it is not true in general that $\mathring{X} \subseteq X \subseteq \overline{X}$.
- $\overline{X}$ and $\mathring{X}$, viewed as mappings on $(2^{\mathcal{X}}, \subseteq)$, are monotone:

$$X \subset X' \Rightarrow \mathring{X} \subseteq \mathring{X}' \text{ and } \overline{X} \subseteq \overline{X}'.$$

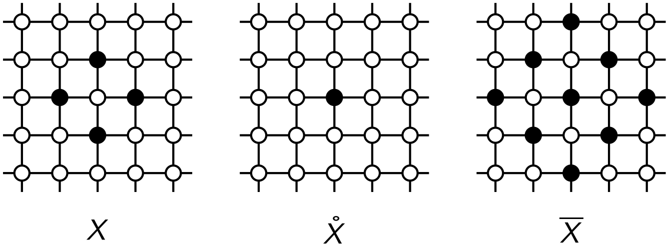


Figure: Interior and closure of a set X

Closure and interior

- We have locally that

$$x \in \mathring{X} \Rightarrow P(x \mid X) = 1, \quad x \notin \overline{X} \Rightarrow P(x \mid X) = 0$$

- Consequently, given that X is the set of active agents at time t, the set X' of active agents at time $t+1$ lies in the interval

$$[\mathring{X}, \overline{X}] := \{Y \in 2^{\mathfrak{X}} \mid \mathring{X} \subseteq Y \subseteq \overline{X}\}.$$

  with probability 1.
- If A is strict and X is finite, then X' can be *any* set in $[\mathring{X}, \overline{X}]$ with a positive probability.

From the previous result, a transition from X is deterministic iff $\mathring{X} = \overline{X}$.
1st case: A *fixed point* is a configuration X such that $\mathring{X} = X = \overline{X}$.

Lemma 1

There is no other fixed point than $\mathcal{X}$ and $\varnothing$.

Deterministic transitions: Blinkers

The other case:

Lemma 2

Consider X such that $\mathring{X} = \overline{X} \neq X$. The following holds.

1. X^c has the same property.
2. $\mathring{X} = X^c$

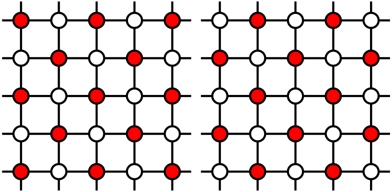
Consequence: for such an X, $\{X, X^c\}$ is a periodic absorbing class of period 2. We call such an X a *blinker*.

Lemma 3

If a blinker exists, it is unique up to complementation.

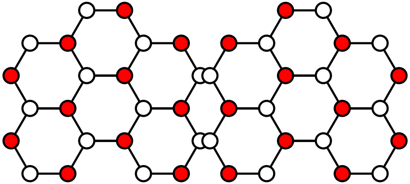
Examples of blinker

- $\mathcal{X} = \mathbb{Z}^2$ with the NESW neighborhood: sum of coordinates is odd



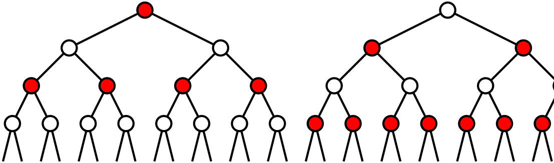
Examples of blinker

- The hexagonal pavement: every 2 nodes on each hexagon



Examples of blinker

- Hierarchies: odd layers



Absorbing and transient sets; irreducibility

- A set $\mathcal{A} \in \mathcal{T}$ is called *absorbing* if $K(\omega, \mathcal{A}) = 1$ for every $\omega \in \mathcal{A}$.
- A set $\mathcal{A} \in \mathcal{T}$ is called *transient* if for every configuration $\omega \in \mathcal{A}$ there exists $n \in \mathbb{N}$ such that $K^n(\omega, \mathcal{A}) < 1$.
- Given ϕ a σ-finite measure on $\mathcal{T}$, the Markov chain is *ϕ-irreducible* if

$$\sum_{n=1}^{+\infty} K^n(\omega, \mathcal{A}) > 0, \forall \omega \in \Omega \text{ whenever } \phi(\mathcal{A}) > 0$$

  i.e., if there is a positive probability that starting from any configuration, the process reaches after some step any set of configurations, provided this set has a positive measure (w.r.t. ϕ).
- A set of configurations $\mathcal{A} \in \mathcal{T}$ is a *ϕ-irreducible set* if for every $\omega \in \mathcal{A}$, every $\mathcal{B} \in \mathcal{T}_{|\mathcal{A}}$ s.t. $\phi_{\mathcal{A}}(\mathcal{B}) > 0$, $K^n(\omega, \mathcal{B}) > 0$ for some n.
- Observe that $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$ are ϕ-irreducible classes (absorbing and ϕ-irreducible sets).

Outline

Partition of Ω

- As a blinker X exists, we partition $\mathcal{X}$ in two blocks, corresponding to X and X^c. We denote this partition as $\mathcal{X} = \mathcal{X}_e \cup \mathcal{X}_o$.
- $\mathcal{X}_e$ is the set of *even nodes*, while $\mathcal{X}_o$ is the set of *odd nodes*.
- We introduce a partition of Ω depending on the number of 1 and 0 in $\mathcal{X}_e$ and $\mathcal{X}_o$.
- A block of the partition is characterized by a 4-uple

$$(a, b, c, d) \text{ with coordinates in } \{0, F, \infty\}$$

  representing the "number" of (even - 0, even - 1, odd - 0, odd - 1).
- There are 25 non-empty blocks in the partition
- Example: $(\infty, 0, \infty, 0)$ is the singleton $\{\mathbf{0}\} = \{\varnothing\}$, $(0, \infty, \infty, 0)$ is the singleton $\{\mathcal{X}_e\}$.

Absorbing and transient sets

Theorem

The following sets of configurations are respectively:

1. *finite ϕ-irreducible classes:*
   - $(\infty, 0, \infty, 0)$,
   - $(0, \infty, 0, \infty)$,
   - $(0, \infty, \infty, 0) \cup (\infty, 0, 0, \infty)$.
2. *infinite uncountable absorbing sets:*
   - $(\infty, \infty, \infty, \infty)$,
   - $(\infty, 0, \infty, \infty) \cup (\infty, \infty, \infty, 0)$,
   - $(0, \infty, \infty, \infty) \cup (\infty, \infty, 0, \infty)$.
3. *infinite transient sets:*
   - (∞, F, ∞, F),
   - (F, ∞, F, ∞),
   - $(F, \infty, \infty, F) \cup (\infty, F, F, \infty)$,
   - $(\infty, F, \infty, \infty) \cup (\infty, \infty, \infty, F)$,
   - $(F, \infty, \infty, \infty) \cup (\infty, \infty, F, \infty)$,
   - $(\infty, F, \infty, 0) \cup (\infty, 0, \infty, F)$,
   - $(F, \infty, 0, \infty) \cup (0, \infty, F, \infty)$,
   - $(0, \infty, \infty, F) \cup (\infty, F, 0, \infty)$,
   - $(F, \infty, \infty, 0) \cup (\infty, 0, F, \infty)$.

ϕ-irreducible sets

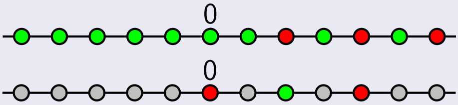
ϕ-irreducibility does not always hold!

Example 1: not enough room to move configurations

Take $\mathbb{Z}$ with the 1-neighborhood. Then $(\infty, \infty, \infty, 0) \cup (\infty, 0, \infty, \infty)$ is not ϕ-irreducible.

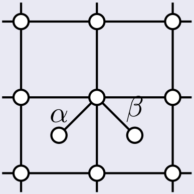
From

one cannot reach



Example 2: not enough room to store configurations

Take $\mathbb{Z}^2$ with the 1-neighborhhod, with two additional nodes α, β. They have same parity, say even. Then again $(\infty, \infty, \infty, 0) \cup (\infty, 0, \infty, \infty)$ is not ϕ-irreducible because α, β having only one neighbor who is common to both, cannot take different statuses.



Complex stars

Definition

A *complex star* is a 7-uple $(s_*, s_1, s_2, s_3, s'_1, s'_2, s'_3) \in \mathcal{X}^7$ such that:

- s_1, s_2, s_3 are 3 distinct nodes;
- $\{s_1, s_2, s_3\} \subseteq \Gamma(s_*)$;
- $s'_1 \in \Gamma(s_1)$, $s'_2 \in \Gamma(s_2)$ and $s'_3 \in \Gamma(s_3)$;
- $s_* \notin \{s'_1, s'_2, s'_3\}$.

Informally, s_* is the center of a star with three branches that have at least a depth of two. Recall that by assumption $(\mathcal{X}, \sim)$ admits a blinker, and therefore we know that $\{s'_1, s'_2, s'_3, s_*\} \cap \{s_1, s_2, s_3\} = \varnothing$. Also, note that we do not assume that s'_1, s'_2 and s'_3 are distinct.

Necessary and sufficient conditions on the graph for the existence of complex stars are known.

Storing configurations

Definition

We say that a partial configuration (X, Y) of $\mathcal{X}_o$ can be *stored* if there exists a mapping θ from $\mathcal{X}_o$ to $\mathcal{X}_e$ such that

1. for every $x \in X \cup Y$, $\theta(x) \in \Gamma(x)$,
2. for every $x \in X$ and every $y \in Y$, $\theta(x) \neq \theta(y)$.

θ is called a *storing function*. Observe that $\theta(X \cup Y) \subseteq \overline{X \cup Y}$.

- When θ is an injection from $X \cup Y$ to $\overline{X \cup Y}$, the problem amounts to finding a matching in the bipartite graph $(X \cup Y, \overline{X \cup Y})$.
- If $(\mathfrak{X}, \sim)$ is k-regular or is a hierarchy, then any configuration can be stored.

Richness Assumption. $(\mathcal{X}, \sim)$ *is said to be rich if:*

1. *There exists an infinite number of complex stars.*
2. *Any partial configuration* (X, Y) *on* $\mathcal{X}_o$ *and on* $\mathcal{X}_e$ *can be stored.*

All the networks introduced in the examples satisfy the richness assumption (except $(\mathbb{Z}^d, \sim)$ with $d = 1$ which has no complex star), since they are all k-regular or a hierarchy, and they contain infinitely many complex stars.

The main result

Theorem

Assume that $(\mathcal{X}, \sim)$ satisfies the Richness Assumption. The following sets are

1. *Finite ϕ-irreducible classes:*
   - **(1)** $(0, \infty, 0, \infty)$ *(this is $\mathcal{X}$);*
   - (2) $(\infty, 0, \infty, 0)$ *(this is $\varnothing$);*
   - **(3)** $(0, \infty, \infty, 0) \cup (\infty, 0, 0, \infty)$ *(this is the blinker);*
2. *Infinite (uncountable) ϕ-irreducible classes:*
   - **(4)** $(\infty, \infty, \infty, \infty)$*;*
   - **(5)** $(\infty, 0, \infty, \infty) \cup (\infty, \infty, \infty, 0)$*;*
   - (6) $(0, \infty, \infty, \infty) \cup (\infty, \infty, 0, \infty)$*;*
3. *Transient and ϕ-irreducible sets:*
   - **(a)** (∞, F, ∞, F)*;*
   - (b) (F, ∞, F, ∞)*;*
   - **(c)** $(F, \infty, \infty, F) \cup (\infty, F, F, \infty)$*;*
   - **(d)** $(\infty, F, \infty, \infty) \cup (\infty, \infty, \infty, F)$*;*
   - (e) $(F, \infty, \infty, \infty) \cup (\infty, \infty, F, \infty)$*;*
   - **(f)** $(\infty, F, \infty, 0) \cup (\infty, 0, \infty, F)$*;*
   - (g) $(F, \infty, 0, \infty) \cup (0, \infty, F, \infty)$*;*
   - **(h)** $(0, \infty, \infty, F) \cup (\infty, F, 0, \infty)$*;*
   - (i) $(F, \infty, \infty, 0) \cup (\infty, 0, F, \infty)$*.*

Interpretation

- Every set of configurations with a "F" is transient: no group of agents can remain active for ever, no cycle between groups, etc.: no polarization is possible.
- Class (3) is a cycle, and classes (5) and (6) are periodic, all of them of period 2.
- An initial finite set of active agents must finish in the long run into one of the absorbing classes (1) to (6). Our study does not permit to say in which one with which probability (seems to be extremely difficult to determine)
- We conjecture however that most probably the process will end in Class (4). In this class, the diffusion is erratic but homogeneous, in the sense that everywhere there are active and inactive agents, on odd and even positions.

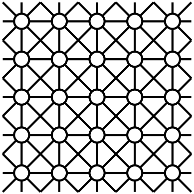
Outline

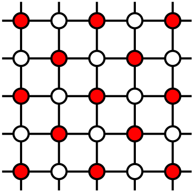
Surviving without blinker

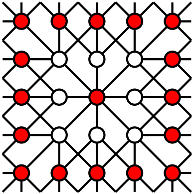
The main proof mechanism still works because of the following result:

Proposition

Given a network $(\mathcal{X}, \sim)$, *there exists a subnetwork* $(\mathcal{X}, \approx)$ *that admits a blinker.*







Results

- As a graph admits in general many subgraphs with a blinker, the distinction between odd and even nodes does not make sense any more.
- Therefore, blocks of the partition of the set of configurations are denoted by (a, b) with $a, b \in \{0, F, \infty\}$, with a, b indicating the number of inactive and active nodes, respectively.

Theorem

Assume that there exists $(\mathcal{X}, \approx)$ a subnetwork of $(\mathcal{X}, \sim)$ with a blinker that satisfies the Richness Assumption. We have the following decomposition:

- $(\infty, 0)$ *and* $(0, \infty)$ *are finite* ϕ*-irreducible classes,*
- (∞, ∞) *is an infinite* ϕ*-irreducible class,*
- (∞, F) *and* (F, ∞) *are transient and* ϕ*-irreducible sets.*

Outline

Reformulating the aggregation function

- The contagion model becomes deterministic.
- An anonymous monotonic aggregation function is necessarily of the form

$$A(1_{\Gamma(x)\cap X}) = \begin{cases} 1, & \text{if } \frac{|\Gamma(x)\cap X|}{\gamma} \geq q \\ 0, & \text{otherwise} \end{cases}$$

  for some $q \in (0,1)$.
- This yields the contagion model of Morris (2000), where the rule of contagion with threshold $0 \leq q \leq 1$ is the following:
  - Given a configuration $X(t)$ at time t, next configuration $X(t+1)$ is the set of agents having a proportion of neighbors in $X(t)$ at least equal to q:

$$X(t+1) = \left\{ x \in \mathcal{X} \, : \, \frac{|\Gamma(x) \cap X(t)|}{|\Gamma(x)|} \geq q \right\}$$

  - The *contagion threshold* ξ is the largest q such that infection spreads over $\mathcal{X}$ from some finite group $X(0)$. Morris (2000) shows that for any network, $\xi \leq \frac{1}{2}$.

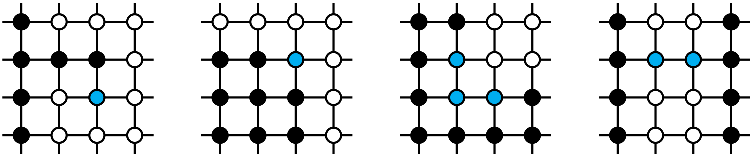
The case of 2-dim mesh with 4 neighbors

Let $X \subseteq \mathcal{X}$, its frontier points are those elements x in X that have outside and interior neighbors.

Let $\eta(x) := |\Gamma(x) \cap X|$ number of interior neighbors of x

Remarkable configurations (from left to right, X in black or blue):

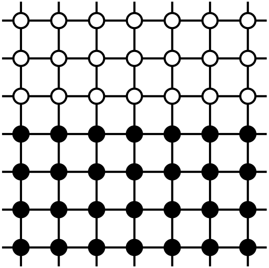
- antenna (x such that $\eta(x) = 1$ in blue)
- convex corner (x such that $\eta(x) = 2$ in blue)
- concave corner in blue
- isthm (x such that $\eta(x) = 2$ in blue)

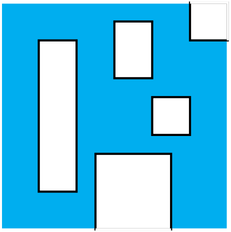


Nontrivial absorbing states

- Only $q = \frac{1}{2}$ or $\frac{3}{4}$ lead to non-trivial absorbing states (Morris 2000).
- X is an absorbing state for $q = \frac{1}{2}$ if and only if $\mathcal{X} \setminus X$ is a possible absorbing state for $q = \frac{3}{4}$.
- For $q = \frac{3}{4}$, each connected component of X should be
  - of size at least 4,
  - with no convex corner, no antenna and no isthm

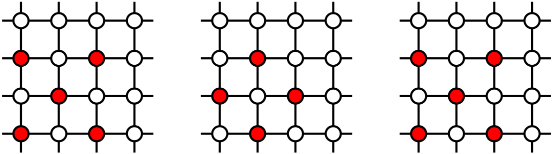
  while each connected component of $\mathcal{X} \setminus X$ should be
  - of size at least 3,
  - and have no antennas.
- Example for $q = \frac{3}{4}$





Other absorbing classes: cycles (periodic trajectories)

Example of a cycle: the 2-dim mesh with 4 neighbors ($q = \frac{1}{2}$)



etc.

Proposition 6

Let A be anonymous and Boolean. Then absorbing classes are either

- singletons $\{X\}$, where $X \in 2^{\mathcal{X}}$,
- cycles (periodic trajectories) of nonempty pairwise incomparable sets $\{X_1, \ldots, X_k\}$

Outline

1. The model
2. General results
3. Strict aggregation functions
4. Boolean anonymous aggregation functions
5. **Conclusion**

Conclusion

- How the diffusion evolves from a (finite) set of active agents depends on the aggregation function for some properties and on the structure of the network for others.
- Transience/persistence of a state relies only on the type of diffusion mechanism, i.e., the aggregation function, without any condition on the network.
- Irreducibility (going from one configuration to another one inside a class) is closely related to the structure of the graph. We have proposed a mild sufficient condition on the structure (richness assumption) to obtain irreducibility.

Conclusion

- We clearly establish a distinction between the probabilistic and the deterministic mechanism (Morris 2000).
- With strict aggregation functions (probabilistic model), no polarization can occur: all finite configurations fade out.
- By contrast, the deterministic model allows the appearance of stable finite or infinite sets of active/inactive agents, that is, polarization can appear, and under many different forms.
- May we conclude that hesitation is a remedy against polarization?