Column-positivity of random labeled networks is resilient and scalable

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Workshop on
Resilient Control of Infrastructure Networks

POLITECNICO
DI TORINO

26th September 2019

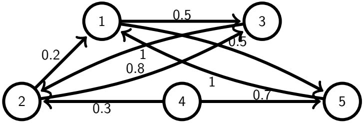
Motivation: discrete-time consensus systems

n agents, x_0 vector of their initial opinions. A = stochastic matrix.

$$\begin{cases} x_{k+1} = Ax_k \\ x_0 \in \mathbb{R}^n_{\geq 0} \end{cases}$$

At **each time** agent i updates her opinion by averaging on the opinions of her neighbours with weights $A[i,:]$.

$$A = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0.5 \\ 0.2 & 0 & 0.8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



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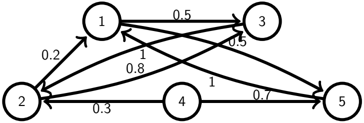
$$A^4 = \begin{pmatrix} 0.25 & 0.65 & 0.5 & 0 & 0.5 \\ 0.2 & 0.64 & 0.21 & 0 & 0.13 \\ 0.26 & 1 & 0.64 & 0 & 0 \\ 0.42 & 0.38 & 0.19 & 0 & 0 \\ 1 & 0 & 0.65 & 0 & 0.25 \end{pmatrix}$$



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The system **converges to consensus** independently on x_0 iff $\mathbf{A^k}$ has an **entrywise positive column** for some $k \in \mathbb{N}$ (equiv. the graph associated to A has a globally reachable node).

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$$\mathcal{M} = \underbrace{\{M_1, \ldots, M_m\}}_{\text{stochastic}}$$

$$\begin{cases} x_{k+1} = M_{i_k} x_k & M_{i_k} \in \mathcal{M} \\ x_0 \in \mathbb{R}^n_{\geq 0} \end{cases}$$

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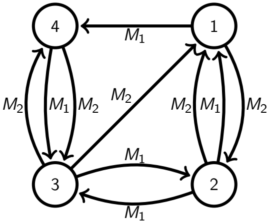

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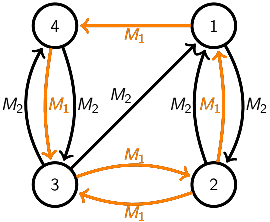

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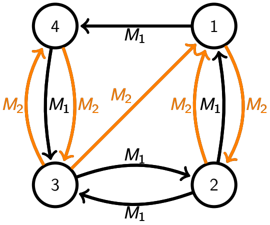

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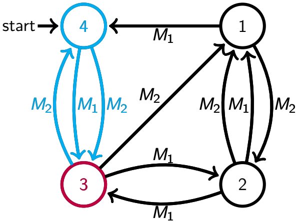

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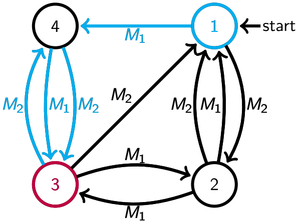

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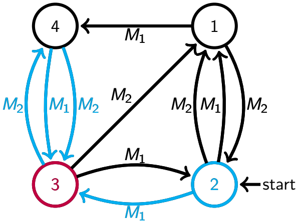

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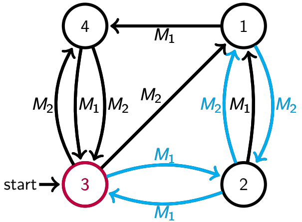

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YES iff there exists a finite product $M_{j_1} \cdot \ldots \cdot M_{j_l}$ of matrices in $\mathcal{M}$ that has an **entrywise positive column**.

Equivalently: iff there exists a vertex that is reachable from any vertex by a path labeled by $M_{j_1}, \ldots, M_{j_l}$.

$\rightsquigarrow$ the length of the shortest positive-column product **influences** the **rate of convergence** to consensus

Column-positivity and pc-index

$$\mathcal{M} = \left\{ M_1 = \begin{pmatrix} 0&0&0&1 \\ 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}, M_2 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 1&0&0&1 \\ 0&0&1&0 \end{pmatrix} \right\}, \; M_1 M_2 M_2 M_1 = \begin{pmatrix} 0&0&1&1 \\ 1&1&1&1 \\ 1&0&1&0 \\ 1&1&1&0 \end{pmatrix} > 0$$

A set $\mathcal{M}$ of nonnegative matrices is **column-positive** if it admits a product with an entrywise positive column. The length of the shortest of these products is called its **pc-index** $[pc(\mathcal{M})]$.

Equivalently the labeled directed graph induced by $\mathcal{M}$ admits a sequence of labels $l = l_1, \ldots, l_s$ and a vertex and that is **reachable** from any other vertex by following a path labeled by l.

Randomized generation of the matrices: $\mathcal{B}_m(n,p)$

$m = \#$ of matrices, $n =$ matrix size

$$\begin{pmatrix} 1 & & \\ & & \end{pmatrix}, \qquad \begin{cases} = 1 & \text{with probability } p = p(n) \\ = 0 & \text{with probability } 1 - p \end{cases}$$

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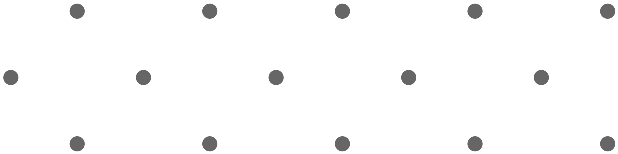

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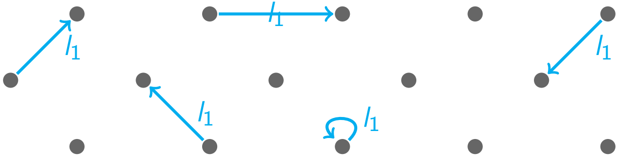

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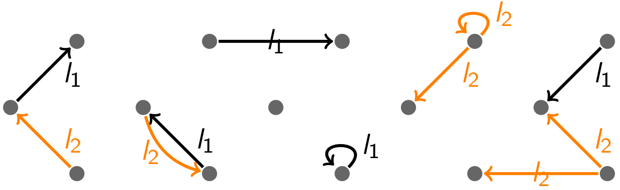

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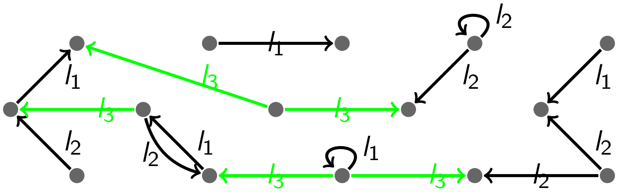

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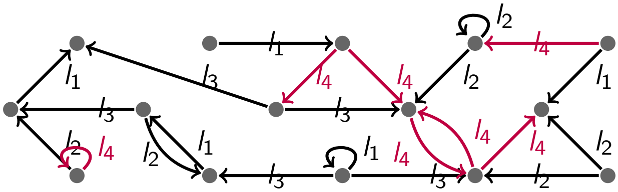

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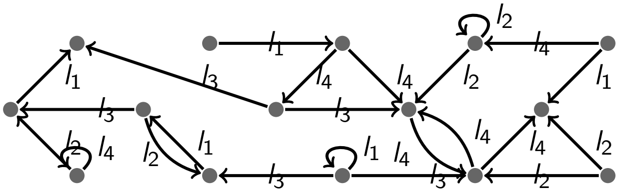

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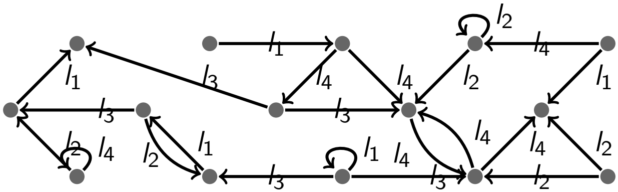


What is the **probability** that $\mathcal{B}_m(n,p)$ is column-positive as $n \to \infty$?
What is its **expected pc-index**?

Sharp threshold for column-positivity

Theorem. [C., Jungers '18] Let $m \geq 2$, $c \in \mathbb{R}$, $\hat{\mathbf{p}} = (\log \mathbf{n} + \mathbf{c})/\mathbf{n}$. For any $p(n) \in [0,1]$, as $n \to \infty$ it holds that:

$$\mathbb{P}\big(\mathcal{B}_m(n,p) \text{ is column-positive}\big) \longrightarrow \begin{cases} 1 & \text{if } \lim_{n\to\infty} \dfrac{p}{\hat{p}} > 1 \\ 0 & \text{if } \lim_{n\to\infty} \dfrac{p}{\hat{p}} < 1 \end{cases}$$

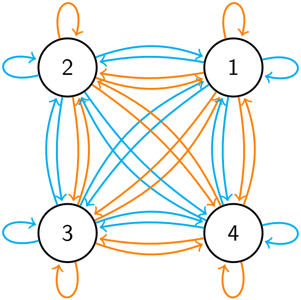
* $\lim_{n\to\infty} \mathbb{P}\big(\mathcal{B}_m(n,\hat{p}) \text{ is column-positive}\big) \leq 1 - \left(1 - e^{-e^{-c}}\right)^m$
* $\lim_{n\to\infty} \mathbb{P}\big(\mathcal{B}_m(n,\hat{p}) \text{ is column-positive}\big) \geq 1 - \left(1 - e^{-2e^{-c}}\right)^m m e^{-2e^{-c}} \left(1 - e^{-2e^{-c}}\right)^{m-1}$

Moreover, if $\lim_{n\to\infty} p/\hat{p} > 1$:

$$\mathbb{P}\big(\mathcal{B}_m(n,p) \text{ is column-positive}\big) \geq 1 - n^{-1} - O\left(ne^{-np}\right) - O\left(n^{-2}\right) .$$

Why resilience?

Complete labeled graph where each edge has probability $\mathbf{1-p}$ to **break**:



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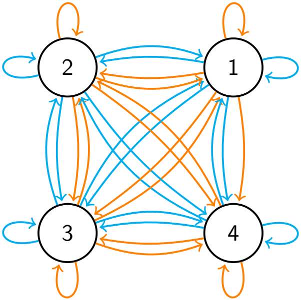

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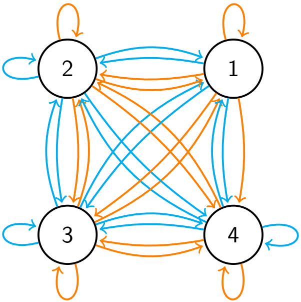

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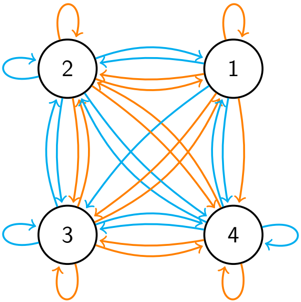

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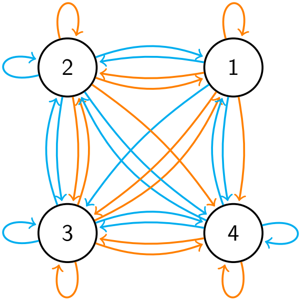

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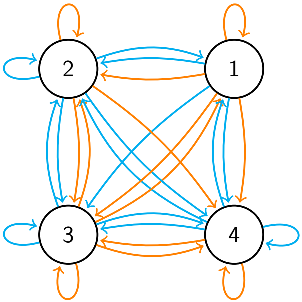

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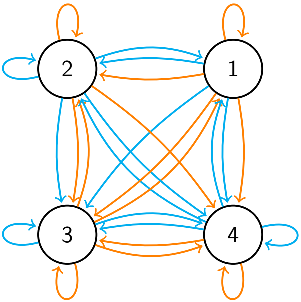

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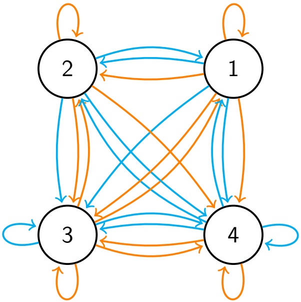

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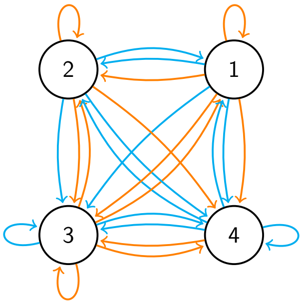

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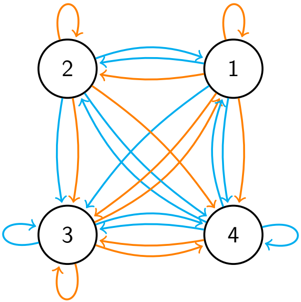

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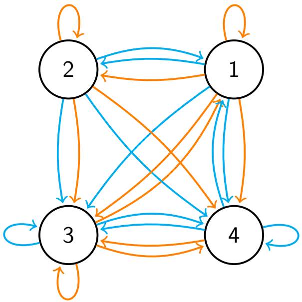

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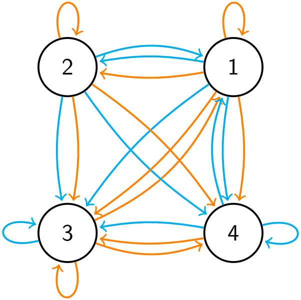

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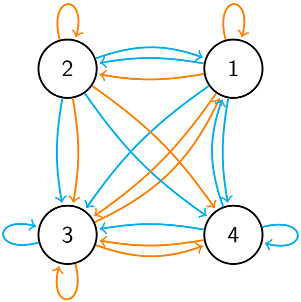

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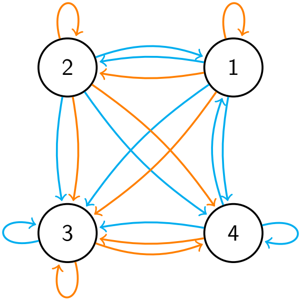
Complete labeled graph where each edge has probability $\mathbf{1-p}$ to **break**:

Does there still exist a node reachable from everywhere by the same sequence of labels?



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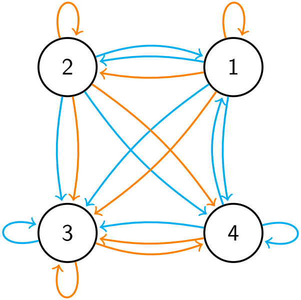

YES when $p > (\log n + c)/n$ (with high probability).
In these terms the network is **resilient** with respect to the **column-positivity property.**

pc-index and scalability

Theorem. [C., Jungers '18] Let $m \geq 2$, $c \in \mathbb{R}$, $\hat{\mathbf{p}} = (\log \mathbf{n} + \mathbf{c})/\mathbf{n}$. Then

* If $\lim_{n\to\infty} p/\hat{p} > 1$: $\lim_{n\to\infty} \mathbb{P}\Big(pc(\mathcal{B}_m(n,p)) = O(n \log n)\Big) = 1$
* If $\lim_{n\to\infty} p/\hat{p} = 1$:

$$\lim_{n\to\infty} \mathbb{P}\Big(pc(\mathcal{B}_m(n,p)) = O(n \log^3 n) \,\Big|\, \mathcal{B}_m(n,p) \text{ has no zero rows}\Big) = 1$$

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The **lenght** of the shortest positive-column product of $\mathcal{B}_m(p)$ is **scaling well** with respect to n with high probability.

$\rightsquigarrow$ The convergence rate to consensus of the consensus system whose matrices are generated by $\mathcal{B}_m(p)$ is **scaling well** with respect to n.

Sketch of the proofs

1. Case $\lim_{n\to\infty} p/\hat{p} > 1$: We prove that $\mathcal{B}_m(n,p) \geq \bar{\mathcal{P}}_m(n)$ whp, where $\bar{\mathcal{P}}_m(n)$ is a set of m permutation matrices with a positive entry added.

- Perfect matchings on random bipartite graphs;
- Prove that if $\bar{\mathcal{P}}_m(n)$ is uniformly distributed, then it is column-positive and with pc-index of $O(n \log n)$ with high probability;
- Few technical lemmas to connect the distributions of $\mathcal{B}_m(n,p)$ and $\bar{\mathcal{P}}_m(n)$.

2. Case $\lim_{n\to\infty} p/\hat{p} < 1$: each matrix of $\mathcal{B}_m(n,p)$ has a zero row whp.

3. Case $p \sim \hat{p} = (\log n + c)/n$:

- Similarly as case 1., but $\bar{\mathcal{P}}_m(n)$ is replaced by a set of $\{0,1\}$-matrices with exactly one 1 in every row (synchronizing automaton);
- Results on random synchronizing automata [Nicaud 2014].

Thank you!