On stability of users equilibria in heterogeneous routing games

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Resilient control of infrastructure networks

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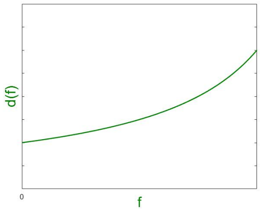
Homogeneous routing games

- Transportation network $\rightarrow$ directed multigraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$
- Behaviour of agents $\rightarrow$ game theory

Notation

- Origin-destination (o, d) $\rightarrow$ route set $\mathcal{R}$
- Route flows z induce unique edge flows via $f_e = \sum_{r:e\in r} z_r$
- $\forall e \in \mathcal{E} \rightarrow$ nondecreasing delay function $d_e(f_e)$
- Cost of route r: $c_r(z) = \sum_{e:e\in r} d_e(f_e)$



Homogeneous routing games

Wardrop equilibrium [Wardrop, 1952]

Wardrop equilibrium $\iff$ admissible route flows s.t.

$$z_r > 0 \;\Rightarrow\; c_r(z) \leq c_q(z) \quad \forall r, q \in \mathcal{R}.$$

Homogeneous games are potential [Beckmann, 1956]

f^* Wardrop equilibrium $\iff$ f^* solves

$$\begin{aligned} \underset{f}{\text{minimize}} \quad & \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(s) ds \\ \text{subject to} \quad & f \geq 0, Bf = \nu. \end{aligned}$$

- existence of the equilibrium
- if delay strictly increasing $\implies$ uniqueness of the equilibrium
- convergence of learning dynamics

Heterogeneous routing games

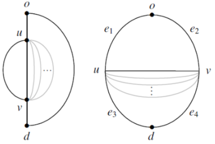
- P populations $\to$ flows f^p $\to$ aggregated flows $f = \sum_p f^p$
- delay functions $d_e^p(f_e)$

Wardrop equilibrium

Wardrop equilibrium $\iff$ admissible route flows s.t.

$$z_r^p > 0 \;\Rightarrow\; c_r^p(z) \leq c_q^p(z) \quad \forall r, q \in \mathcal{R}.$$

- Heterogeneous games are not potential
  - Existence of equilibrium [Schmeidler, 1973]
  - Essential uniqueness (i.e. uniqueness of the aggregated flows) for series of nearly parallel graphs [Milchtaich, 2005]
  - Open question: what about learning dynamics?



Logit dynamics

$$\dot{z}_r^p = \tau^p \cdot \frac{\exp\Big(-\eta \cdot c_r^p(\sum_{q=1}^{P} z^q)\Big)}{\sum_{s\in\mathcal{R}} \exp\Big(-\eta \cdot c_s^p(\sum_{q=1}^{p} z^q)\Big)} - z_r^p \quad \forall p \in \mathcal{P}, \forall r \in \mathcal{R}.$$

- η inverse of noise level
- τ^p throughput of population p

- $\eta = 0$: divergent noise, randomization

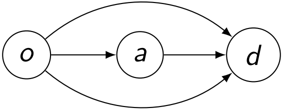
$$\dot{z}_r^p = \tau^p \cdot \frac{1}{|\mathcal{R}|} - z_r^p$$

- $\eta \to \infty$: no noise, best response dynamics

Dynamics on simple graphs

Simple graph

A graph is simple if every edge belongs to one route only.



Logit dynamics on simple graphs

$$\dot{f}_e^p = \tau^p \cdot \frac{\exp\left(-\eta \cdot d_e^p(\sum_{q=1}^{P} f_e^q)\right)}{\sum_{j=1}^{L} \exp\left(-\eta \cdot d_j^p(\sum_{q=1}^{P} f_j^q)\right)} - f_e^p \quad \forall p \in \mathcal{P}, \forall e \in \mathcal{E}$$

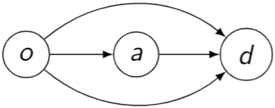
Main result: global convergence on (series of) simple graphs

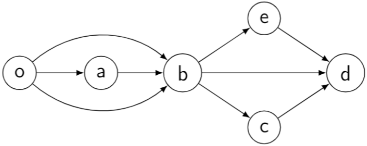
Theorem

- *$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ series of simple directed graphs.*
- *P populations, with non-decreasing delay functions $d_e^p(f_e)$*

Then:

1. *logit dynamics converges to a globally asymptotically stable equilibrium $f^*(\eta)$*
2. $\lim_{\eta \to +\infty} f^*(\eta)$ *approaches the set of the Wardrop equilibria*





Sketch of the proof

For simple graphs the dynamics of the aggregated flows reads

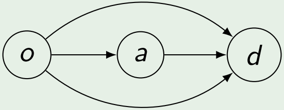
$$\dot{f}_e = \sum_{p=1}^{P} [\tau^p \cdot \frac{\exp\left(-\eta \cdot d_e^p(f_e)\right)}{\sum_{j=1}^{L} \exp\left(-\eta \cdot d_j^p(f_j)\right)}] - f_e \quad \forall e \in \mathcal{E}$$

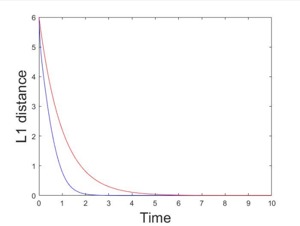
- The evolution of f is autonomous.
- J is Metzler and strictly diagonally dominant by columns.

$\implies$ l_1 contraction [Como et al., 2015]

- For series of simple graphs we prove in cascade

Simulations: simple graphs



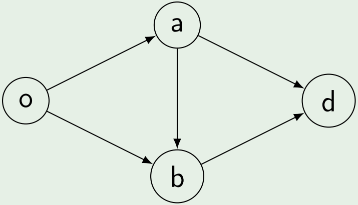


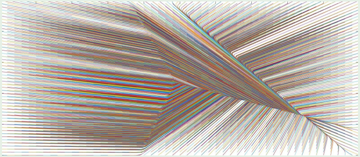
$$\|f(t) - g(t)\|_{l_1} \leq \|f(0) - g(0)\|_{l_1} e^{-t}$$

Remark

- Simple $\Rightarrow$ nearly parallel
- Nearly parallel $\not\Rightarrow$ simple

Simulations: nearly parallel graphs



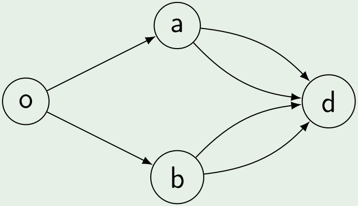


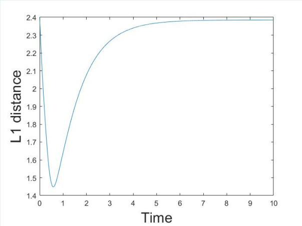
$$P = 2 \quad \eta = 10$$

Conjecture

Globally asymptotically stable equilibrium on nearly parallel graphs

Simulations: not nearly parallel graphs





$P = 2 \quad \eta = 10$

- Bifurcation point
  - $\eta \to \infty$: two attractive equilibria
  - $\eta = 0$: globally asymptotically stable equilibrium

Conclusions

- Stability of the equilibria under logit dynamics in (series of) simple graphs

- Conjecture: globally asymptotically stable equilibrium even in (series of) nearly parallel graphs

- Bifurcation in graphs with multiple equilibria
    - $\eta = 0$: unique equilibrium (randomization)
    - $\eta \to \infty$: multiple stable equilibria

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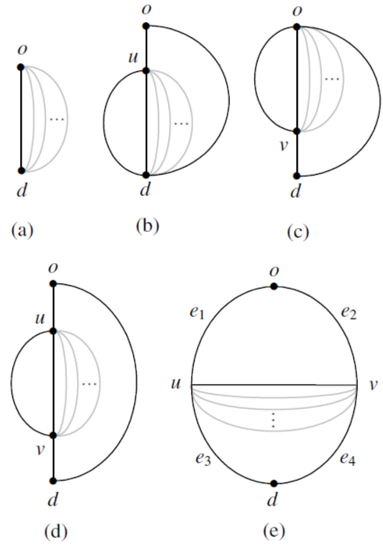
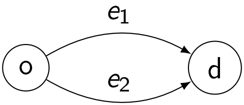
Nearly parallel





If it exists $\tilde{f}$ s.t.

$$d_1^1(\tilde{f}) = d_2^1(\tau - \tilde{f}), \quad d_1^2(\tilde{f}) = d_2^2(\tau - \tilde{f}), \tag{1}$$

all the edge flows satisfying

$$\begin{gathered} f_1^1 + f_1^2 = \tilde{f}, \quad f_2^1 + f_2^2 = \tau - \tilde{f}, \\ f_1^1 + f_2^1 = \tau^1, \quad f_1^2 + f_2^2 = \tau^2, \end{gathered} \tag{2}$$

are Wardrop equilibria.

$$d_1^1(f_1) = f_1 + 1, \quad d_1^2(f_1) = 2f_1, \quad \tau^1 = 1;$$
$$d_2^1(f_2) = 2f_2, \quad d_2^2(f_2) = f_2 + 1; \quad \tau^2 = 1.$$