On the convergence of linear uncertain systems

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Workshop – Resilient Control of Infrastructure Networks
September 24–27th, 2019 – Turin, Italy

State convergence is a quintessential problem

- G. Belgioioso, F. Fabiani, F. Blanchini and S. Grammatico, *On the convergence of discrete-time linear systems: a linear time-varying Mann iteration converges iff its operator is strictly pseudocontractive*, L-CSS, 2018.
- F. Fabiani, G. Belgioioso, F. Blanchini, P. Colaneri and S. Grammatico, *Convergence in uncertain linear systems*, arXiv, 2019.

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Problem

Convergence of the state variables to a constant, a-priori unknown state

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Theoretical problems:

- Consensus
- Optimization & game theory
- Multi-agent learning
- ...

Applications:

- Power systems
- Social networks
- Robotic and sensor networks
- ...

Why uncertain systems?

Polar opinion dynamics in social networks

N-Agents in a social network, opinion vector $x \in [-1, 1]^N$

$$\dot{x} = -A(x)Lx$$

- $L \in \mathbb{R}^{N \times N}$ Laplacian matrix of the communication graph
- $A(x) \in \mathrm{diag}\left([0,1]^N\right)$ susceptibility to *persuasion*

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$$A(x) \in \mathrm{conv}\{A_1, \ldots, A_m\} \text{ for all } x \in [-1,1]^N$$

Always exist functions $\alpha_i(t,x) \geq 0$, $\sum_{i\in\mathcal{M}} \alpha_i(t,x) = 1$ s.t.

$$\dot{x} = -\left(\sum_{i\in\mathcal{M}} \alpha_i(t,x) A_i L\right) x$$

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General framework and definitions

Difference (differential) inclusion with polytopic uncertainty:

$$x(k+1) = \sum_{i \in \mathcal{M}} A_i w_i(k) x(k) \coloneqq A(w(k))x(k) \tag{1}$$

with $w(k) \in \mathcal{W} \coloneqq \left\{ w \in \mathbb{R}^m \,\middle|\, \sum_{i \in \mathcal{M}} w_i = 1,\ w_i \geq 0\ \forall i \in \mathcal{M} \right\}.$

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$$x(k) \to \bar{x} \text{ strongly depends on } \{w(k)\}_{k \in \mathbb{N}}$$

Example:

$$A(w(k)) = \begin{bmatrix} a_{1,1}(k) & 0 \\ 1 & 1 \end{bmatrix}, \quad a_{1,1}(k) \in \{1/2, 3/4\}\ \forall k \in \mathbb{N}$$

$$\begin{cases} x_1(k) \to 0 \\ x_2(k) \to \overline{x}_2 = x_2(0) + \sum_{k \geq 0} x_1(k) \end{cases}$$

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Definition

The system in (1) is weakly convergent if, $\forall w(k) \in \mathcal{W}$ and $\forall x(0) \in \mathbb{R}^n$, $\exists \bar{x} \in \mathbb{R}^n$ s.t. $\lim_{k \to \infty} \| x(k) - \bar{x} \| = 0$.

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The system in (1) is strongly convergent if, $\forall w(k) \in \mathcal{W}$ and $\forall x(0) \in \mathbb{R}^n$, $\exists \bar{x} \in \bar{\mathcal{X}} := \bigcap_{i \in \mathcal{M}} \ker(A_i - I)$ s.t. $\lim_{k \to \infty} \| x(k) - \bar{x} \| = 0$.

The Kernel Sharing Property (KSP)

When weak convergence implies strong convergence

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$\{A_i\}_{i \in \mathcal{M}}$ has the KSP if $\ker(A_i - I) = \bigcap_{i \in \mathcal{M}} \ker(A_i - I)$, $\forall i \in \mathcal{M}$.

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Proposition

If the family of matrices $\{A_i\}_{i\in\mathcal{M}}$ has the KSP, then weak convergence $\Longrightarrow$ strong convergence.

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2. $\Longrightarrow$ $\ker(A(w) - I)$ is invariant w.r.t. $w \in \mathcal{W}$

Theorem

$x^+ \in A(w)x$ strongly convergent $\Longleftrightarrow$ $\exists\ T \in \mathbb{R}^{n\times n}$ s.t., for all $i \in \mathcal{M}$,

$$T^{-1}A_iT = \begin{bmatrix} A_i^{\mathrm{as}} & 0 \\ A_i^{\mathrm{r}} & I_m \end{bmatrix}$$

$m \coloneqq \dim(\bar{\mathcal{X}})$, while $\{A_i^{\mathrm{as}}\}_{i\in\mathcal{M}}$ generate a.s. difference inclusion.

Strong convergence

Lyapunov-like results

Theorem

$x^+ \in A(w)x$ *strongly convergent* $\implies$ *existence of wPLF.*

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$x^+ \in A(w)x$ strongly convergent $\implies$ existence of wPLF.

- $x^+ \in A(w)x$ strongly convergent $\;\not\Longrightarrow\;$ existence of wQLF

Example:

Asymptotic stability as strong convergence to the kernel $\{0\}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 + w(t) & -1 \end{bmatrix} x, \quad |w(t)| \leq \rho \; \forall t$$

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- Existence of wPLF or wQLF $\not\Longrightarrow$ strong/weak convergence

Weak convergence

LaSalle arguments for differential inclusions $\dot{x} \in A(w)x$

Insight:

$$x \to \bar{x} \neq 0 \implies \begin{cases} \text{marginal stability} \\ \text{matrix } \bar{A} \in \mathrm{conv}\left(\{A_i\}_{i \in \mathcal{I}}\right) \text{ is singular} \end{cases}$$

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Common kernel not required

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Common kernel not required

Example:

$$A(w(t)) \in \{A_1, A_2\} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\}$$

- if $A(w)$ "converges" to $\bar{A} = \frac{1}{2}(A_1 + A_2)$ (singular) $\Rightarrow x \to [\bar{x}_1; 0]$

Beyond convergence issues...

- Operator-splitting methods for decomposable, large-scale SDPs

$$
\begin{cases}
\min\limits_{X} & \langle C, X \rangle \\
\text{s.t.} & \langle A_k, X \rangle = b_k, \ \forall k \in \mathcal{K}, \\
 & X \in \mathbb{S}^n_+.
\end{cases}
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$$\begin{cases} \min\limits_{X_1,\dots,X_N} & \sum\limits_{i\in\mathcal{I}} \langle C_i, X_i \rangle \\ \text{s.t.} & \sum\limits_{i\in\mathcal{I}} \langle A_{k,i}, X_i \rangle = b_k,\ \forall k \in \mathcal{K}, \\ & E_i X_i E_i^\top - E_j X_j E_j^\top = 0,\ \forall (i,j) \in \mathcal{I}\times\mathcal{N}_i, \\ & X_i \in \mathbb{S}_+^{|\mathcal{C}_i|},\ \forall i \in \mathcal{I}. \end{cases}$$

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- Distributed algorithms for nonconvex games

$$\forall i \in \mathcal{I} : \begin{cases} \min\limits_{x_i \in \mathcal{X}_i} & J_i(x_i, \boldsymbol{x}_{-i}) \\ \text{s.t.} & g(\boldsymbol{x}) \le 0. \end{cases}$$

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1. Nonmonotone VIs
2. Moments approach

Thank you for your kind attention!

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