Resilient Control Workshop

Stochastic Nash Equilibrium Problems

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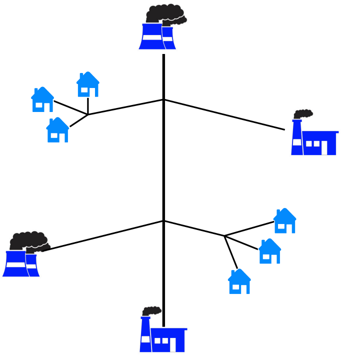
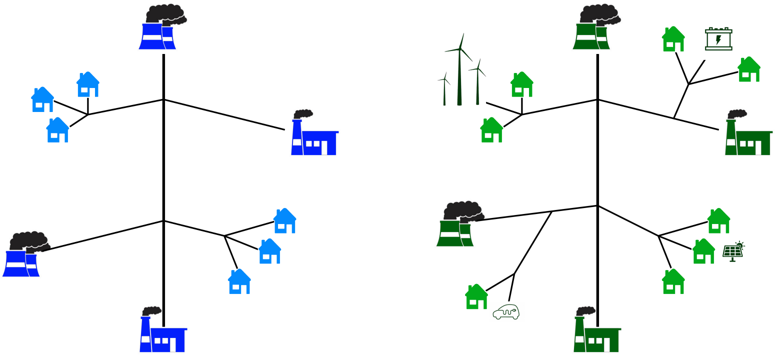
Motivation

Electricity Market



Motivation

Electricity Market



Stochastic Nash Equilibirum Problem

- Set of agents: $\mathcal{N} = \{1, \dots, N\}$
- Decision variable: $x_i \in \mathcal{X}_i$
- Cost Function: $\mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) = \mathbb{E}_\xi[J_i(x_i, \boldsymbol{x}_{-i}, \xi)]$
- Uncertainty: $\xi \in \Xi$ with unknown distribution

Optimization Problem

$$\begin{cases} \min\limits_{x_i \in \Omega_i} & \mathbb{J}_i(x_i, \boldsymbol{x}_{-i}, \xi) = \mathbb{E}_\xi[J(x_i, \boldsymbol{x}_{-i}, \xi)] \\ \text{s.t.} & x_i \in \mathcal{X}_i \end{cases} \quad \forall i \in \mathcal{N}$$

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Assumption

- $\mathbb{J}_i(\cdot, \mathbf{y})$ *is convex and continuously differentiable.*
- $\mathcal{X}_i$ *is nonempty, compact and convex.*
- $J_i(\boldsymbol{x}, \xi)$ *is convex, Lipschitz continuous, and continuously differentiable in* x_i *for each* $\boldsymbol{x}_{-i}$.
  $\xi \mapsto J_i(\boldsymbol{x}, \xi)$ *is measurable and for each* $\boldsymbol{x}_{-i}$.
  The Lipschitz constant $\ell_i(\xi, \boldsymbol{x}_{-i})$ *is integrable in* ξ.

Stochastic Nash Equilibrium

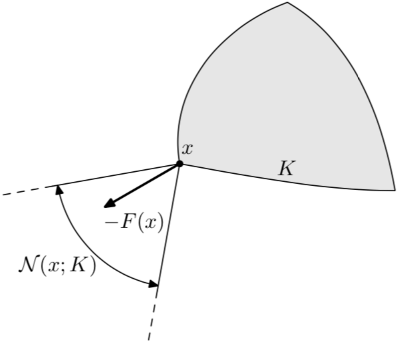
A stochastic Nash equilibrium is a collective strategy $\boldsymbol{x}^* \in \mathcal{X}$ such that for all $i \in \mathcal{N}$

$$\mathbb{J}_i(x_i^*, \boldsymbol{x}_{-i}^*) \leq \inf\{\mathbb{J}_i(y, \boldsymbol{x}_{-i}^*) \mid y \in \mathcal{X}_i\}.$$

U. Ravat and U. V. Shanbhag, On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM Journal on Optimization*, 21(3):1168-1199, 2011.

Stochastic Variational Inequality Problem

Find $x^* \in \mathcal{X}$ such that $\langle F(x), x - x^* \rangle \geq 0$, for any $x \in X$.



Stochastic Variational Inequality Problem

Find $x^* \in \mathcal{X}$ such that $\langle F(x), x - x^* \rangle \geq 0,$ for any $x \in X$.

$$\Rightarrow \quad F(\boldsymbol{x}) = \begin{bmatrix} \mathbb{E}[\nabla_{x_1} J_1(x_1, \boldsymbol{x}_{-1}, \xi)] \\ \vdots \\ \mathbb{E}[\nabla_{x_N} J_N(x_N, \boldsymbol{x}_{-N}, \xi)] \end{bmatrix} = \mathbb{E}[\nabla J(\boldsymbol{x})]$$

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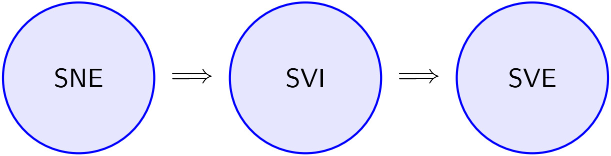
Assumption

F is η-strongly monotone, i.e., there exists $\eta > 0$ such that

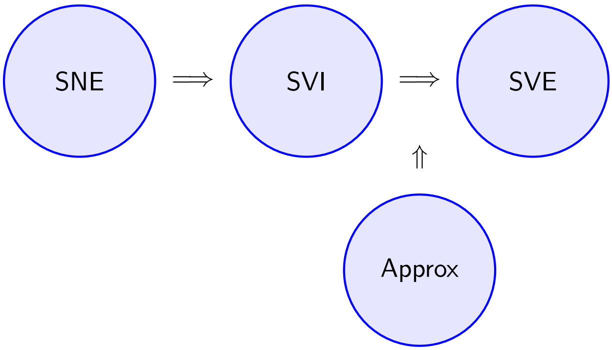
$$\langle F(x) - F(y), x - y \rangle \geq \eta \left\| x - y \right\|^2 \text{ for all } x, y \in \mathbb{R}^n$$

Variational Equilibria



F. Facchinei, J. Pang, Nash Equilibria: The Variational Approach, Convex Optimization in Signal Processing and Communication, *Cambridge University Press*, Cambridge, 2009, pp. 443-493.

Variational Equilibria



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Approximation Schemes

Stochastic Approximation

$$\tilde{F}(\boldsymbol{x}, \xi) = \begin{bmatrix} \nabla J_1(\boldsymbol{x}, \xi_1) \\ \vdots \\ \nabla J_N(\boldsymbol{x}, \xi_N) \end{bmatrix}$$

Approximation Schemes

Stochastic Approximation

$$\tilde{F}(\boldsymbol{x}, \xi) = \begin{bmatrix} \nabla J_1(\boldsymbol{x}, \xi_1) \\ \vdots \\ \nabla J_N(\boldsymbol{x}, \xi_N) \end{bmatrix}$$

Sample average

$$F_{avg}(\boldsymbol{x}, \xi) = \frac{1}{N} \sum_{k=1}^{N} \nabla J(\boldsymbol{x}, \xi^{(k)})$$

STOCHASTIC GENERALIZED NASH EQUILIBRIUM PROBLEMS

- Set of agents: $\mathcal{N} = \{1, \dots, N\}$
- Cost Function: $\mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) = \mathbb{E}_\xi[J_i(x_i, \boldsymbol{x}_{-i}, \xi)]$
- Uncertainty: $\xi \in \Xi$ with unknown distribution

- Shared Constraints: $x_i \in \mathcal{X}(\boldsymbol{x}_{-i})$

$$\mathcal{X}_i(\boldsymbol{x}_{-i}) := \{y_i \in \Omega_i \mid A_i y_i \leq b - \sum_{j \neq i}^{N} A_j x_j\},$$

Stochastic Generalized Nash Equilibrium Problems

- Set of agents: $\mathcal{N} = \{1, \dots, N\}$
- Cost Function: $\mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) = \mathbb{E}_\xi[J_i(x_i, \boldsymbol{x}_{-i}, \xi)]$
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$$\Rightarrow \quad \begin{cases} \min\limits_{x_i \in \Omega_i} & \mathbb{J}_i(x_i, \boldsymbol{x}_{-i}, \xi) = \mathbb{E}_\xi[J_i(x_i, \boldsymbol{x}_{-i}, \xi)] \\ \text{s.t.} & A\boldsymbol{x} \leq b \end{cases} \quad \forall i \in \mathcal{N}$$

Stochastic Generalized Nash Equilibrium Problems

- Lagrangian function

$$\mathcal{L}_i\left(\boldsymbol{x}, \lambda_i\right) := \mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) + \lambda_i^\top (A\boldsymbol{x} - b)$$

- Karush-Khun-Tucker Conditions of the game

$$\begin{cases} 0 \in \mathbb{E}[\nabla_i J_i(x_i, \boldsymbol{x}_{-i})] + A_i^\top \lambda_i, \\ \lambda_i \geq 0, (\lambda_i)^\top (A\boldsymbol{x} - b) = 0, \\ A\boldsymbol{x} - b \leq 0 \end{cases}$$

Stochastic Variational Inequality

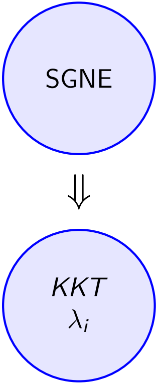
- x^* is a solution of SVI$(\mathcal{X}, F)$ if and only if

$$x^* \in \operatorname{argmin}_{y\in\mathcal{X}} (y - x^*)^\top F(x^*),$$

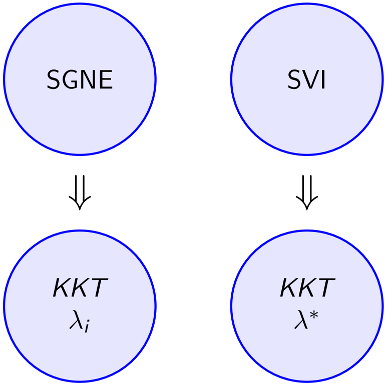
- Karush-Khun-Tucker of the VI

$$\begin{cases} 0 \in \mathbb{E}[\nabla J(\boldsymbol{x})] + A_i^\top \mu, \\ \mu \geq 0, \mu^\top (A\boldsymbol{x} - b) = 0, \\ A\boldsymbol{x} - b \leq 0 \end{cases}. \tag{1}$$

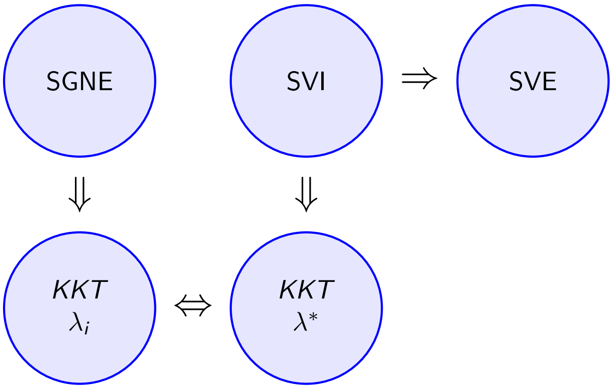
Variational Equilibria



Variational Equilibria



Variational Equilibria



F. Facchinei, A. Fischer, V. Piccialli. On generalized Nash games and variational inequalities. *Operations Research Letters*, 35(2):159-164, 2007.

Thank You for Your Attention!