LUNDS UNIVERSITET

# An Estimation Error in a Distributed System

**Sept. 2019**

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# The System and The Policy

- We start with a discrete time, linear system

$$x_{k+1} = ax_k + u_k + w_k,$$

where the value of $a$ is not known to the controller

- We rearrange terms to get, for any $k$,

$$a = \frac{\sum_{l<k}(x_{l+1} - u_l)x_l}{\sum_{l<k} x_l^2} - \frac{\sum_{l<k} w_l x_l}{\sum_{l<k} x_l^2}$$

- With this estimate, $a_k$, we set $u_k = -a_k x_k$, and get

$$x_{k+1} = (a - a_k)x_k + w_k = e_k x_k + w_k$$

- This policy typically works well, both with random noise, with systematic noise and with modelling errors.

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# Adversarial Disturbance

- For any value $\gamma$, and with $a = 0$, there is a sequence $w_k$ such that

$$\sum_k |x_k|^2 > \gamma^2 \sum_k |w_k|^2$$

- Despite unbounded $l_2$-gain, there is BIBO stability, i.e., with bounds on $a$ and the disturbance, we can bound the effects the disturbance

$$\sum_k |x_k|^2 < \Phi(a, \sum_k |w_k|^2)$$

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# The Essential Argument

- For our system

$$x_{k+1} = e_k x_k + w_k$$

with, for all $k$,

$$\sum_{l<k} w_l^2 < \sigma^2 + h^2 \sum_{l<k} x_l^2$$

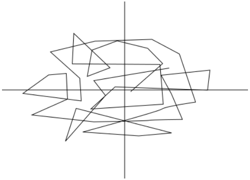
- There is a bound on $|e_k|$ that will always hold eventually
- There may be a phase of initial confusion, but we will always reach a state of knowledge about $a$, and the impact of confusion can be bounded.
- This enables a small-gain argument handling model error.
- The question is now: what happens for $x \in \mathbb{R}^n$

# The First Problem

- When, $x_k \in \mathbb{R}^n$ and $w_k \in \mathbb{R}^n$, $a \in \mathbb{R}^{n \times n}$, with estimates for $a$ and bounds on the magnitude of $w_k$, we again have

$$x_{k+1} = e_k x_k + w_k$$

- Our $x$ has to be spread out, to explore the dynamics in $n$ dimensions, but the adversarial $w$, can focus all its magnitude along a single dimension



Penalty factor $\sqrt{n}$

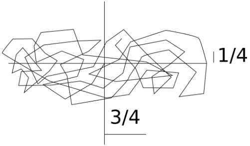
- Acceptable for $n = 3$, but difficult to use for $n = 100$

# Sparse Connectivity

- We consider a distributed system with many nodes that are sparsely connected.
- For a linear chain of nodes, connected to neighbours, the matrix $a$ is tridiagonal.
- In this case, we impose a bound on $w_k$ with an interpretation of local modelling error, and we use local (row-wise) error estimates for our dynamical matrix $a$.
- Best case penalty factor becomes $\sqrt{3}$ instead of $\sqrt{n}$, and it depends on connectivity of each node instead of on the number of nodes.

# The Remaining Problem

- The penalty factor is not actually $\sqrt{n}$ or $\sqrt{3}$
- With unexplored directions, the penalty factor can be arbitrarily large, even for $n = 2$




Penalty factor $\sqrt{4}$ instead of $\sqrt{2}$

- It depends on the least explored direction in state space

$$\min_{|\eta|=1} \sum_k \left(\eta \cdot x_k\right)^2$$

- To make progress, I would need a lower bound on the "evenness" of exploration of different state-space directions.

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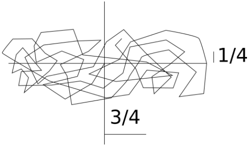


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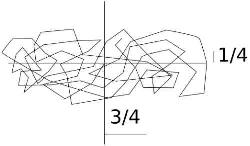


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# The Problem with our Method

- We did not impose any relationship between $w_k$ and $x_k$.
- We said: Given a sequence $x_k$, find a sequence $w_k$ that maximizes $|e_k|$.
- If we write out the full dynamics explicitly, we find that $x_{k+1}$ is a linear combination of $w_1, \ldots, w_k$

$$x_{k+1} = w_k - \sum_{l<k} c_{kl} w_l$$

- Even more explicitly

$$x_{k+1} = w_k - \sum_{l<k} \left[ \sum_{\eta} \frac{(\eta \cdot x_k)(\eta \cdot x_l)}{\sum_{l'<k} (\eta \cdot x_{l'})^2} \right] w_l$$

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# The End

Thank you!