Dipartimento di Scienze Matematiche G. L. Lagrange

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Systemic risk and network intervention

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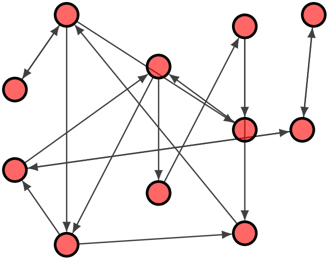
Network Dynamics in the Social, Economic, and Financial sciences

POLITECNICO DI TORINO

Introductory example



Variables of interest

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}) \longrightarrow$ Directed network

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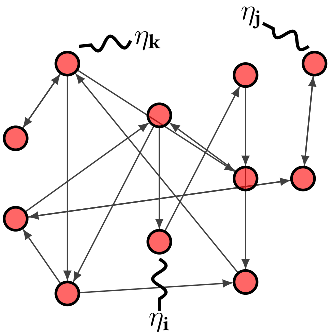

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- η in $\mathbb{R}^n \longrightarrow$ Shocks

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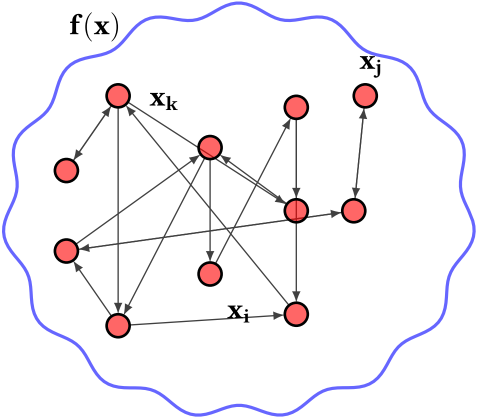

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- $\mathcal{G} = (\mathcal{V}, \mathcal{E}) \longrightarrow$ Directed network
- η in $\mathbb{R}^n \longrightarrow$ Shocks
- x in $\mathbb{R}^n \longrightarrow$ Agents' equilibrium state
- $f(x)$ in $\mathbb{R} \longrightarrow$ Aggregate observable

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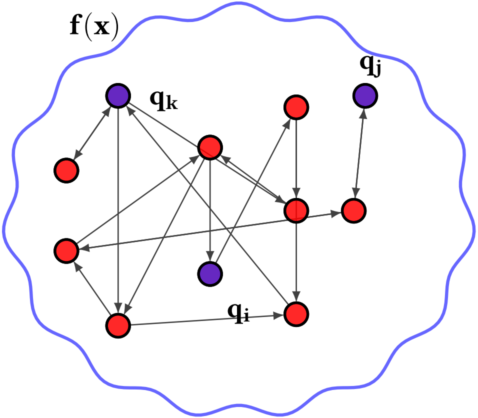

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- q in $\mathbb{R}^n \longrightarrow$ Protection

Formalization of the problem

Equilibria configuration

$$x = (I - D\,P)^{-1}(I - D)\,c \qquad (1)$$

P in $\mathbb{R}^{n\times n}$, s.t. $P \geq 0$ and $P\mathbb{1} = \mathbb{1}$

D in $\mathbb{R}^{n\times n}$, diagonal such that $0 < D_{ii} < 1,\ \forall i$.

c in $\mathbb{R}^n \longrightarrow c = \bar{c} + Q^{-1}\eta$ where

- $\bar{c}$ reference vector;
- η **shock** vector, a random vector $\mathbb{E}[\eta] = 0$ and $\mathrm{Cov}(\eta) = \Omega$;
- $Q = \mathrm{diag}(q)$, q is the **protection** vector, $q_i \geq 1, \forall i$.

Optimal protection against shocks

We studied **effect of shocks and relative protections** in the equilibrium configuration given a budget constraint C:

$$\min_{q_i \geq 1,\, \|q\| \leq C} \; \max_{\mathrm{Tr}(\Omega) \leq 1} \sum_i \mathrm{Var}[x_i]. \tag{W}$$

It is particularly relevant to study also the sample mean

$$\min_{q_i \geq 1, \|q\| \leq C} \; \max_{\mathrm{Tr}(\Omega) \leq 1} \mathrm{Var}[n^{-1} \mathbb{1}' x]. \tag{M}$$

Useful notation

- $L = (I - DP)^{-1}(I - D)$: interaction matrix, Leontief matrix, etc.
- $v = n^{-1} L' \mathbb{1}$: Katz-Bonacich centrality vector
- $\ell_i = \|L_{\cdot i}\|_2$: euclidean norm of L's columns

Application (I)

Our starting ideas: analysis of shocks on economic and production network [Acemoglu2010],[Acemoglu2012], and [Carvalho2014].

Production network model

$$x = \alpha P x + (1-\alpha)c, \quad \alpha \in (0,1)$$

$$y = \log(GDP) = n^{-1}\mathbb{1}' x$$

- The components of the vector c have the meaning of **marginal benefits** of the economic agents
- **Micro** shocks $\longrightarrow$ **macro** fluctuations
- $\mathrm{Var}(y)$ is called **aggregate volatility**

Application (II)

We are given a set of players $\mathcal{V}$ whose utilities are given by

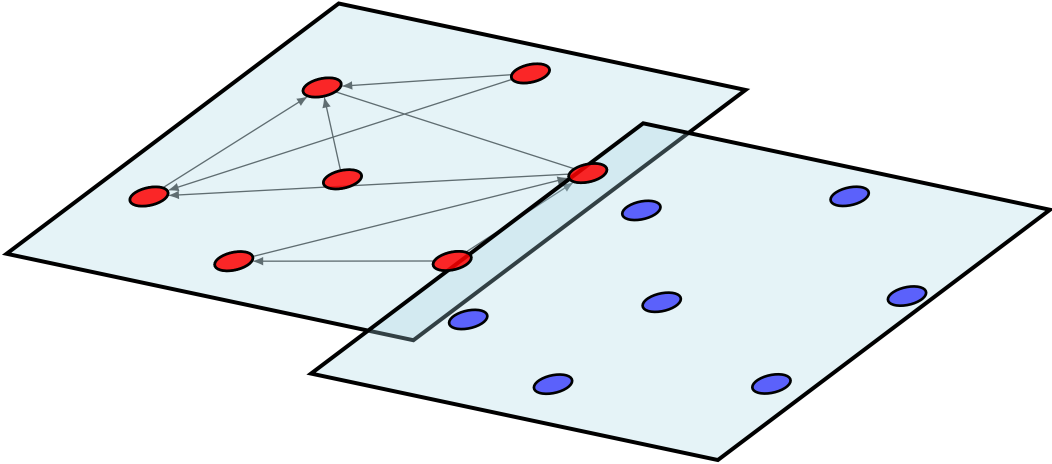
Coordination games and network quadratic games

$$u_i(x) = -\frac{1}{2}\left[\sum_j W_{ij}(x_i - x_j)^2 + \rho_i(x_i - c_i)^2\right]$$

Large literature in coordination games and quadratic network games [Ballester2016], [Galeotti2010],[Bramoulle2014],and [Galeotti2017].

- In the sociological models the vector c represents the **initial opinion** of the agents
- In the network intervention context it represents **standalone marginal return**

The dependence graph



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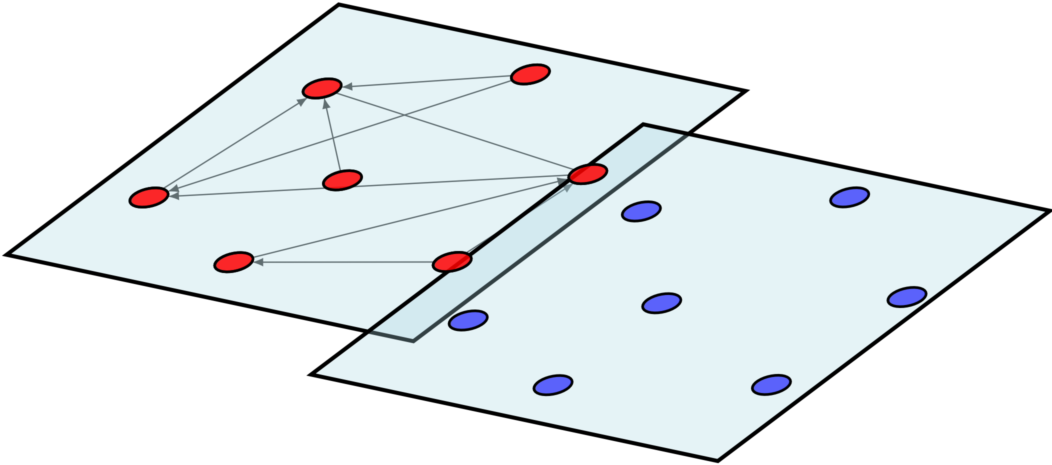
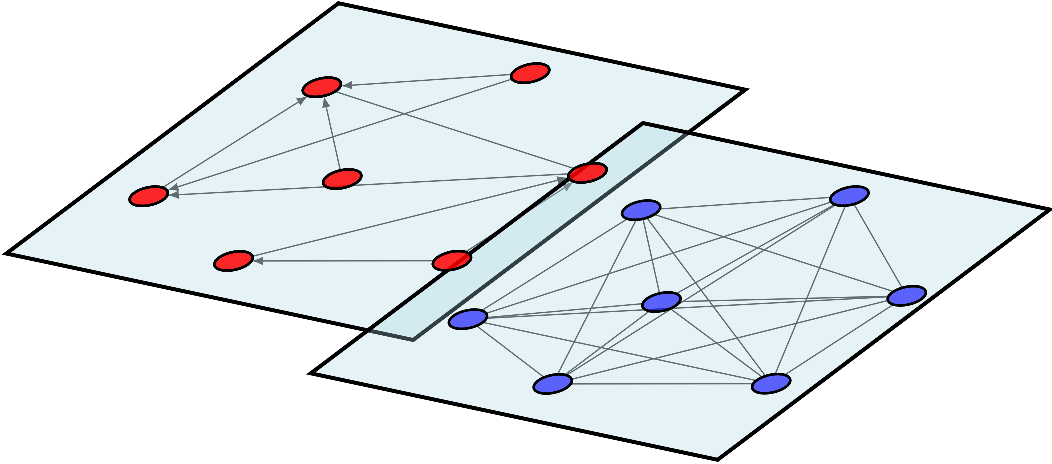


- Independent shocks $\longrightarrow$ isolated nodes

The dependence graph



- Independent shocks $\longrightarrow$ isolated nodes
- Correlated shocks $\longrightarrow$ complete graph

Independent shocks

$$\Omega = \operatorname{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2) \longrightarrow \begin{cases} (W) & \min\limits_{q_i \ge 1, \|q\| \le C} \max\limits_{\|\sigma\| \le 1} \sum\limits_i \left(\sigma_i \dfrac{\ell_i}{q_i}\right)^2 \\ (M) & \min\limits_{q_i \ge 1, \|q\| \le C} \max\limits_{\|\sigma\| \le 1} \sum\limits_i \left(\sigma_i \dfrac{v_i}{q_i}\right)^2 \end{cases} .$$

Consider y and assume $y_1 \geq y_2 \geq \dots \geq y_n$

Proposition

Solution of max problem is

$$\max_{\|\sigma\| \le 1} \sum_i (\sigma_i y_i / q_i)^2 = \max_i (y_i / q_i)^2$$

given by every $\sigma : \sigma_j = \begin{cases} k_j \in (0,1), & j \in K = \{ j \in \mathcal{V} : (y_j/q_j)^2 = \|y/q\|_\infty^2 \} \\ 0, & j \notin K \end{cases}$,

and such that $\sum\limits_{j \in K} k_j^2 = 1$.

Introduce the function:

$$f(\lambda) = \sum_{i=1}^{n} \max\left\{1, \left(y_i/\sqrt{\lambda}\right)\right\} \qquad \lim_{\lambda \to 0^+} f(\lambda) = +\infty, \quad f(y_1^2) = n$$
$$C \geq \sqrt{n} \Longrightarrow \lambda(C) := f^{-1}(C^2).$$

Let $k(C)$ be the maximum index such that $y_{k(C)} > \lambda(C)^{1/2}$.

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The initial problem becomes

$$\min_{q_i \geq 1, \|q\| \leq C} \max_i \left(y_i/q_i\right)^2 .$$

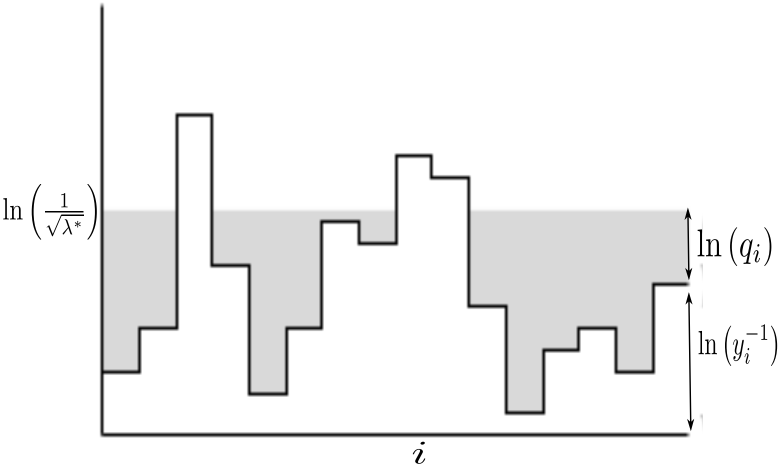
Proposition

The optimum is

$$\lambda(C) = \left(C^2 - (n - k(C))\right)^{-1} \sum_{i}^{k(C)} y_i^2$$

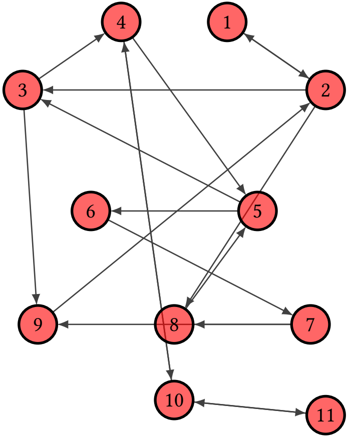
and is reached by $\quad q_i = \max\left\{1, y_i/\sqrt{\lambda(C)}\right\}, \ \forall i$

Water filling



Initial example

$x = (1 - \alpha)(I - \alpha P)^{-1} c$



$\alpha = 0.8,\ n = 11,\ C = \sqrt{n+1}$

Difference between v and ℓ

i	v	ℓ	q_v	q_ℓ
1	0.0476	0.0628	1.0000	1.0000
2	0.1473	0.1365	**1.3359**	**1.2836**
3	0.0949	0.0837	1.0000	1.0000
4	0.0986	0.0939	1.0000	1.0000
5	0.1183	0.1004	**1.0725**	1.0000
6	0.0655	0.0663	1.0000	1.0000
7	0.0705	0.0748	1.0000	1.0000
8	0.0758	0.0734	1.0000	1.0000
9	0.1138	0.1005	**1.0321**	1.0000
10	0.1061	0.1237	1.0000	**1.1629**
11	0.0606	0.0840	1.0000	1.0000

Correlated shocks

Ω full covariance matrix $\longrightarrow$ correlation between each pair of shocks

$$
\begin{aligned}
(W) \qquad & \min_{q_i \geq 1,\, \|q\| \leq C} \max_{\mathrm{Tr}(\Omega) \leq 1} \mathrm{Tr}\left(Q^{-1} \Omega Q^{-1} L' L\right) \\
(M) \qquad & \min_{q_i \geq 1,\, \|q\| \leq C} \max_{\mathrm{Tr}(\Omega) \leq 1} v' Q^{-1} \Omega Q^{-1} v.
\end{aligned}
$$

Correlated shocks

Ω full covariance matrix $\longrightarrow$ correlation between each pair of shocks

$$(W) \quad \min_{q_i \geq 1,\, \|q\| \leq C} \max_{\mathrm{Tr}(\Omega) \leq 1} \mathrm{Tr}\left(Q^{-1}\Omega Q^{-1} L' L\right)$$

$$(M) \quad \min_{q_i \geq 1,\, \|q\| \leq C} \max_{\mathrm{Tr}(\Omega) \leq 1} v' Q^{-1}\Omega Q^{-1} v.$$

for a symmetric matrix A:

$$\max_{\mathrm{Tr}(\Omega) \leq 1} \mathrm{Tr}(\Omega A) = \|A\|_2$$

$$\Downarrow$$

$$(W) \quad \min_{q_i \geq 1,\, \|q\| \leq C} \sigma_1\left(Q^{-1} L' L Q^{-1}\right)$$

for every vector x:

$$\max_{\mathrm{Tr}(\Omega) \leq 1} x'\Omega x = \|x\|^2$$

$$\Downarrow$$

$$(M) \quad \min_{q_i \geq 1,\, \|q\| \leq C} v' Q^{-2} v$$

min-max for the arithmetic mean

$f(\lambda) = \sum_{i=1}^{n} \max\left\{1, \left(v_i/\sqrt{\lambda}\right)^{1/2}\right\}$

$C \geq \sqrt{n} \Longrightarrow \lambda(C) := f^{-1}(C^2)$

$k(C)$index such that $y_{k(C)} > \lambda(C)^{1/2}$.

Proposition

It holds

$$\min_{q_i \geq 1 \, \|q\| \leq C} \sum_k \left(\frac{v_k}{q_k}\right)^2 = \sum_{k=1}^{k(C)} v_k \sqrt{\lambda(C)} + \sum_{k=k(C)+1}^{n} v_k^2$$

The optimum is reached by

$$q_k = \max\left\{1, \left(v_k/\sqrt{\lambda(C)}\right)^{1/2}\right\}$$

min-max for the sum of variances?

$$\min_{q_i \geq 1,\, \|q\| \leq C} \sigma_1 \left(Q^{-1} L' L Q^{-1} \right)$$

- is a quasi-convex problem (but it is not easy to get explicit solution)
- $\sigma_1 \left(Q^{-1} L' L Q^{-1} \right)$ solves $\det(\sigma_1 Q^2 - L'L) = 0$ and it is known as the **generalized eigenvalue of the pair** $(Q, L'L)$
- simulations show that optimum q has 'water filling' structure

Conclusions

- Characterization of the min-max problem for an equilibrium configuration of the system
- Analysis of the uncorrelated and totally correlated shocks $\longrightarrow$ the nature of the shock is fundamental
- Solutions for low budget present 'water filling' structure
- What does it happen for general dependence relation?

Thank you for the attention

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