POLITECNICO DI TORINO

DISMA Dipartimento di Scienze Matematiche G. L. Lagrange

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On network games with coordinating and anti-coordinating agents

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Joint work with Laura Arditti, Giacomo Como, Fabio Fagnani

Network Dynamics in the Social, Economic, and Financial Sciences

5 November 2019

Motivation

- **Network games**: strategic interactions over interconnected systems
- **Coordinating agents**: spread of social norms and innovations
- **Anti-coordinating agents**: traffic congestion, crowd dispersion and division of labor
- Irregular network topology and population heterogeneity are not sufficient to cause nonexistence of Nash equilibria; **coexistence of coordinating and anti-coordinating agents** must play a role (Ramazi et al, 2016)

Preliminaries

Game: $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i \in \mathcal{V}})$

1. Agent set: $\mathcal{V}$
2. Action set: $\mathcal{A}$
3. Utilities: $u_i : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$, $i \in \mathcal{V}$

Game: $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$

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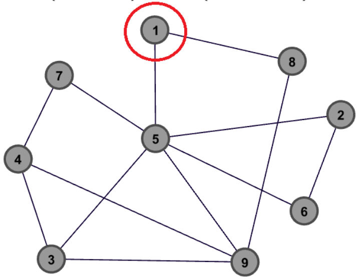
Network game: $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$

- Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
- Utilities depend only on their action and their **neighbors**' actions

$$u_1(x_1, x_{-1}) = u_1(x_1, x_5, x_8)$$



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Best response function:

$$\mathcal{B}_i(x_{-i}) = \arg\max_{x_i\in\mathcal{A}} u_i(x_i, x_{-i})$$

Nash equilibrium:

$$x_i^* \in \mathcal{B}_i(x_{-i}^*) \qquad i \in \mathcal{V}$$

Network game: $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$

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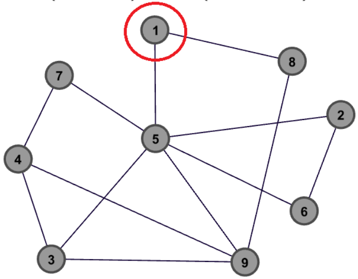

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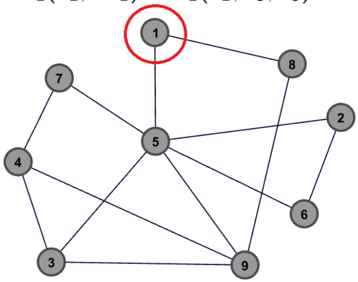

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Potential game: $u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) = \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i})$

$\to$ **Existence of Nash equilibrium guaranteed**

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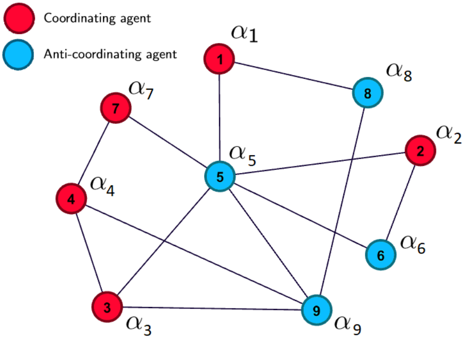
Heterogeneous network coordination anti-coordination game

$$u_i(x_i, x_{-i}) = \begin{cases} \sum_{j\in\mathcal{V}} W_{ij}x_ix_j - \alpha_i x_i & i \text{ coordinating agent} \\ -\sum_{j\in\mathcal{V}} W_{ij}x_ix_j + \alpha_i x_i & i \text{ anti-coordinating agent} \end{cases}$$



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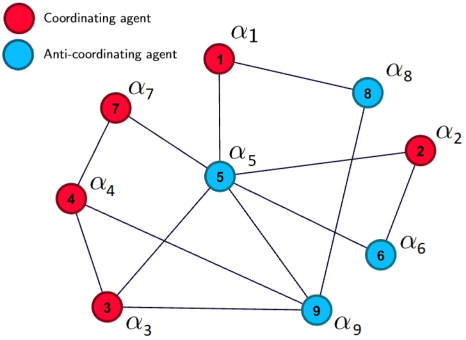

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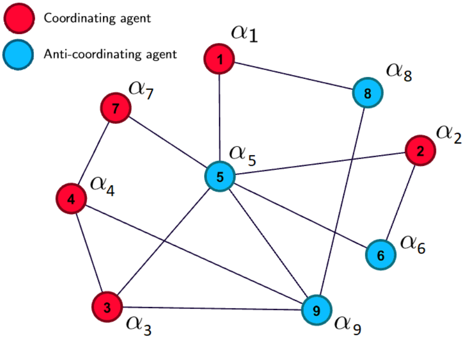
- Undirected $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
- Binary $\mathcal{A} = \{-1, +1\}$



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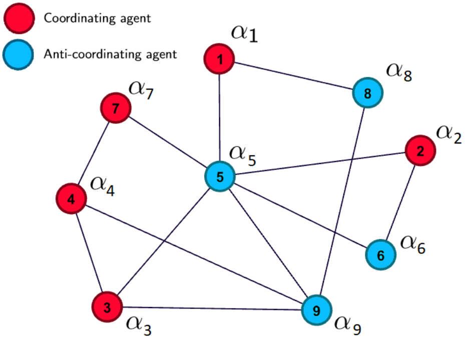
- Undirected $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
- Binary $\mathcal{A} = \{-1, +1\}$
- Node weights $\{\alpha_i\}_{i \in \mathcal{V}}$, $\alpha_i \in \mathbb{R}$



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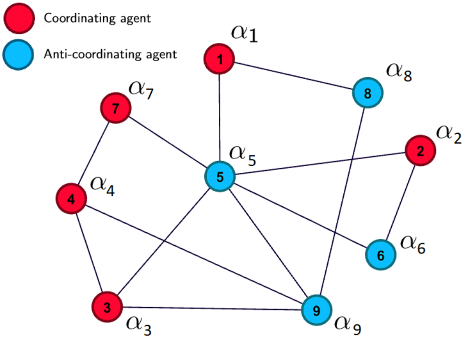
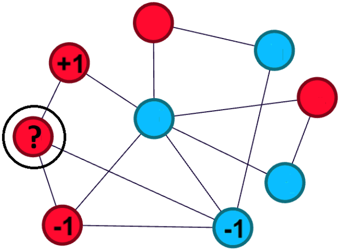
- Binary $\mathcal{A} = \{-1, +1\}$
- Node weights $\{\alpha_i\}_{i \in \mathcal{V}}$, $\alpha_i \in \mathbb{R}$
- Anti-coordinating agents $\mathcal{V}_a$



Best response function

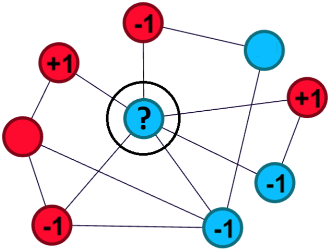
Coordinating agent $i \in \mathcal{V}_c$

$$\mathcal{B}_i(x_{-i}) = \text{sign}(w_i^+(x) - r_i \; w_i)$$



Anti-coordinating agent $i \in \mathcal{V}_a$

$$\mathcal{B}_i(x_{-i}) = -\text{sign}(w_i^+(x) - r_i \; w_i)$$



Coordinating agent $i \in \mathcal{V}_c$

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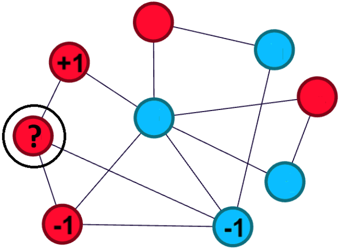

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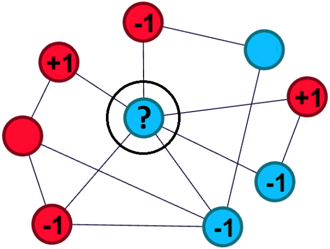

- **Thresholds:** $r_i = \frac{1}{2} + \frac{\alpha_i}{2w_i}$
- $w_i = \sum_{j \in \mathcal{V}} W_{ij}$ (degree), $w_i^+(x) = \sum_{j \in \mathcal{V}} W_{ij} \frac{x_j + 1}{2}$

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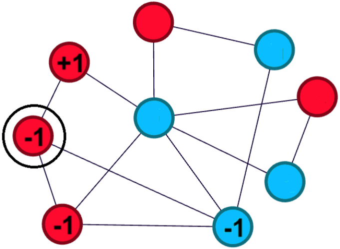

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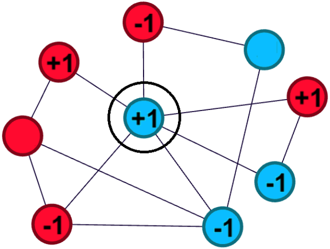

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Network coordination game

Utilities: $$u_i^c(x_i, x_{-i}) = \sum_{j \in \mathcal{V}} W_{ij} x_i x_j - \alpha_i x_i$$

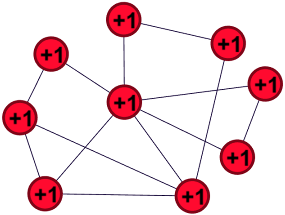
- Symmetric two-player game, undirected graph
  $\rightarrow$ The network game is potential
- Homogeneous thresholds $\rightarrow$ Potential game (straightforward)
- Heterogeneous thresholds?

Proposition

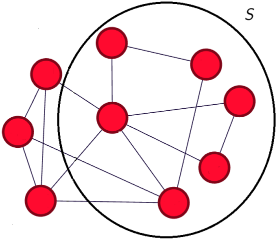
If undirected graph, then ***potential function***

$$\Phi_c(x) = \frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij} x_i x_j - \sum_{i \in \mathcal{V}} \alpha_i x_i$$

- Existence of Nash equilibria guaranteed



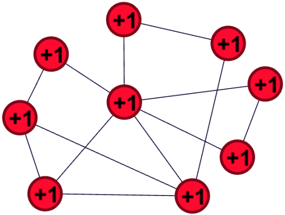
Consensus always Nash equilibrium



$\mathcal{S} \subseteq \mathcal{V}$ **r-cohesive** if $\frac{\sum_{j \in \mathcal{S}} W_{ij}}{w_i} \geq r$ for all $i \in \mathcal{S}$.

Theorem (Morris, 2000)

$x = \mathbb{1}_S - \mathbb{1}_{\mathcal{V} \setminus S}$ *Nash equilibrium* $\Leftrightarrow$ *S is r-cohesive*, *$\mathcal{V} \setminus S$ is $(1-r)$-cohesive*



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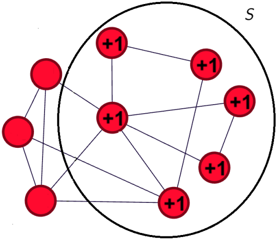

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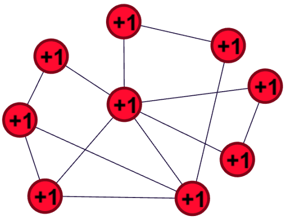

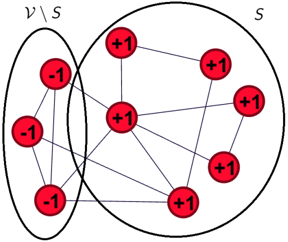

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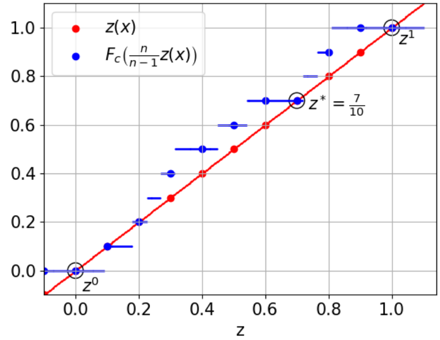
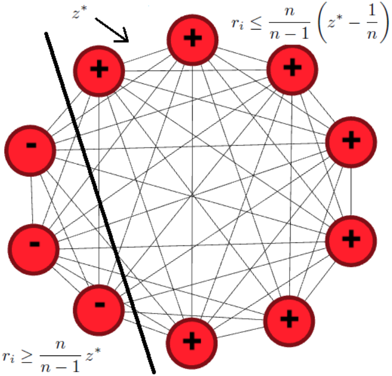
Characterization NE complete graph

$$x^* \in \mathcal{N} \quad \Leftrightarrow \quad z^* = F\left(\frac{n}{n-1}(z^* - \epsilon)\right), \quad \forall \epsilon \in \left(0, \frac{1}{n}\right]$$

where $F(z) := \frac{1}{n}\{i \in \mathcal{V}_c \mid r_i \leq z\}$ and $z^* := \frac{1}{n}\{i \in \mathcal{V} \mid x_i^* = +1\}$.





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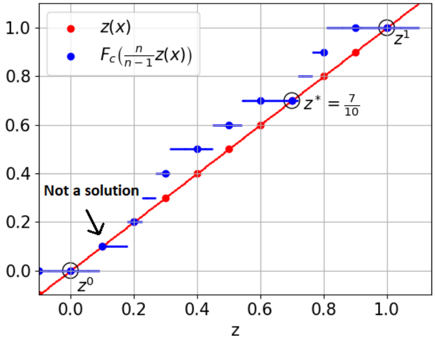
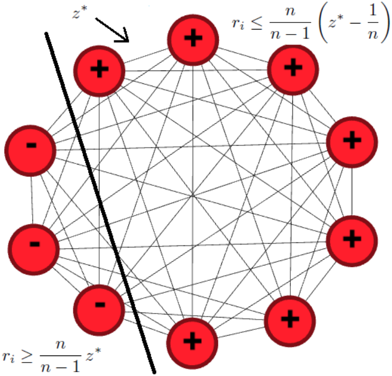




Network anti-coordination game

Utilities: $$u_i^a(x_i, x_{-i}) = -u_i^c(x_i, x_{-i}) = -\sum_{j\in\mathcal{V}} W_{ij}x_ix_j + \alpha_i x_i$$

- Characterization of Nash equilibria not trivial
- Homogeneous thresholds $\rightarrow$ Potential game (straightforward)
- Heterogeneous thresholds?

Proposition

If undirected graph, then ***potential function***

$$\Phi_a(x) = -\Phi_c(x) = -\frac{1}{2}\sum_{i,j\in\mathcal{V}} W_{ij}x_ix_j + \sum_{i\in\mathcal{V}} \alpha_i x_i$$

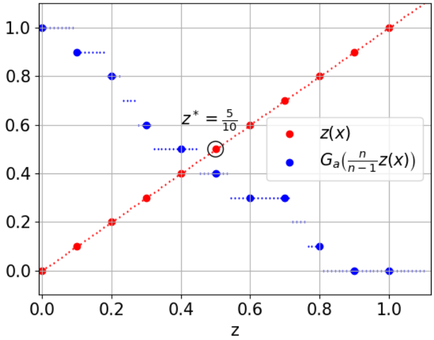
- Existence of Nash equilibria guaranteed over any possible undirected network

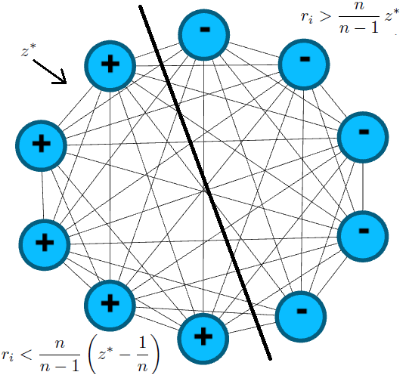
Network anti-coordination game

Characterization NE complete graph

$$x^* \in \mathcal{N} \quad \Leftrightarrow \quad G\left(\frac{n}{n-1}(z^* - \epsilon)\right) \geq z^* \geq G\left(\frac{n}{n-1} z^*\right), \quad \forall \epsilon \in \left(\frac{1}{n}, \frac{2}{n}\right]$$

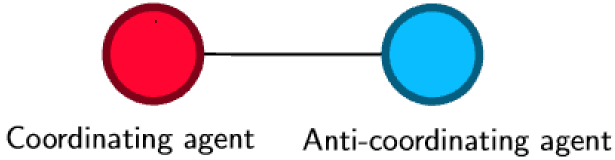
where G(z) = 1-F(z).





Proposition

One ***edge*** *between a coordinating agent and an anti-coordinating agent*
$\rightarrow$ ***not*** *a potential game*



The discoordination game admits no Nash equilibria

Main result

- Undirected $\mathcal{G}$
- $\mathcal{V}_a$ anti-coordinating agents
- $\mathcal{V}_c := \mathcal{V} \setminus \mathcal{V}_a$ coordinating agents
- **Thresholds** $r_i = \frac{1}{2} + \frac{\alpha_i}{2w_i} = r$ for all $i \in \mathcal{V}$

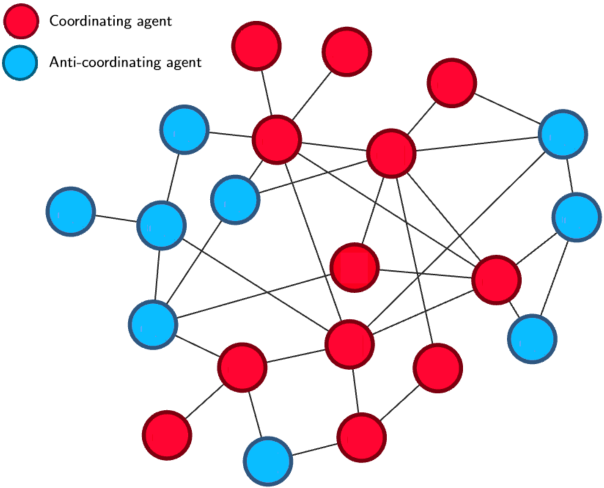
Theorem (Sufficient condition for NE)

*Set of coordinating agents $\mathcal{V}_c$ r-**cohesive** (or $(1-r)$-**cohesive**)*
$\rightarrow$ at least one Nash equilibrium

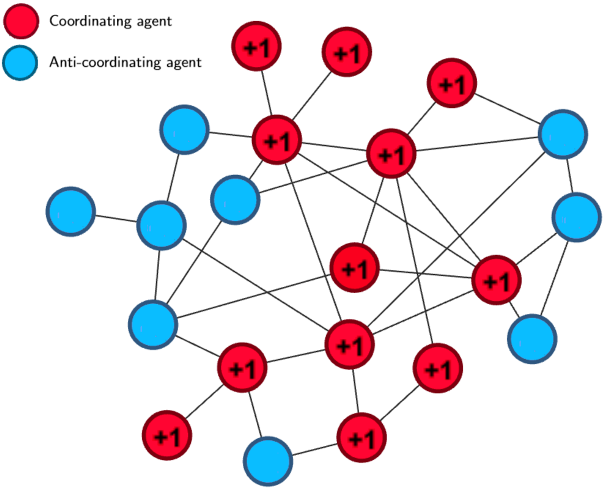
Recall: $\mathcal{S} \subseteq \mathcal{V}$ r-**cohesive** if $\frac{\sum_{j\in\mathcal{S}} W_{ij}}{w_i} \geq r$ for all $i \in \mathcal{S}$.

Outline of the proof

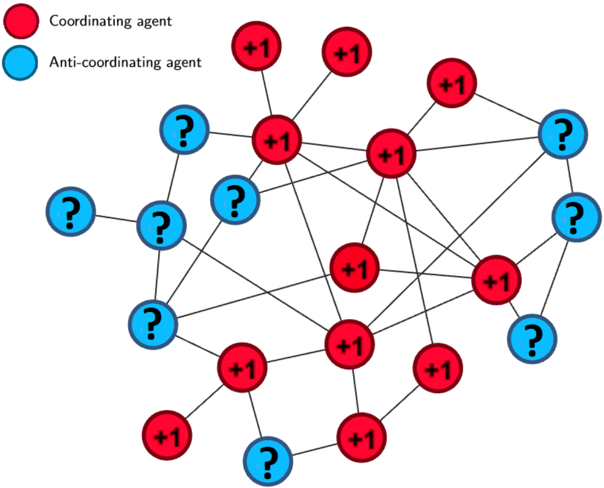
$\mathcal{V}_c$ is $\frac{1}{2}$-cohesive



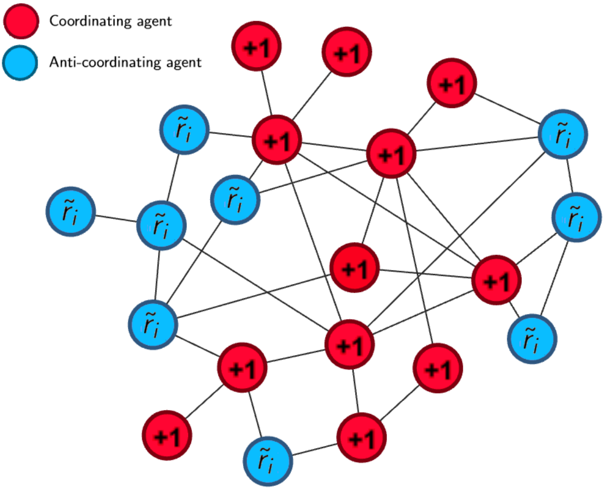
(+1)-stubborn agents



Network anti-coordination game with stubborn agents

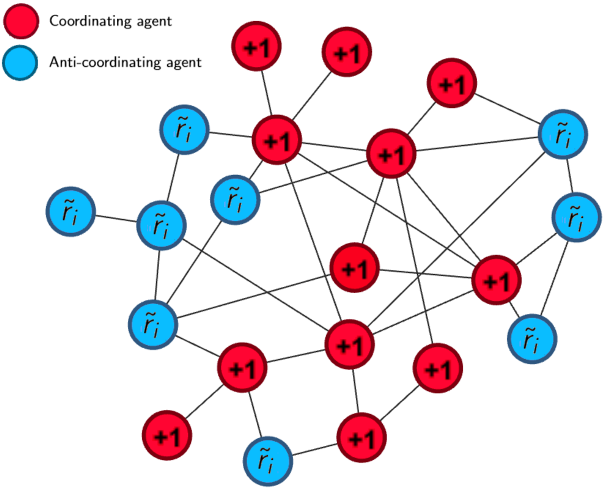


Heterogeneous network anti-coordination game

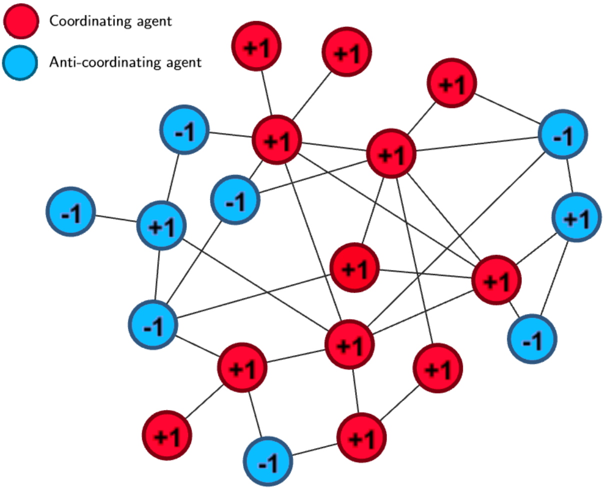


Outline of the proof

Heterogeneous network anti-coordination game $\rightarrow$ Potential game



Nash equilibrium



Nash equilibria on complete graph

Proposition (Sufficient and necessary for NE on the complete graph)

$$x^* \in \mathcal{N} \quad \Leftrightarrow \quad \mathbf{z}^*,\ z_c^*,\ z_a^* \text{ satisfy: } \begin{cases} z_c^* = F_c\left(\frac{n}{n-1}(\mathbf{z}^* - \epsilon_c)\right) \\ G_a\left(\frac{n}{n-1}(\mathbf{z}^* - \epsilon_a)\right) \geq z_a^* \geq G_a\left(\frac{n}{n-1}\mathbf{z}^*\right) \\ \mathbf{z}^* = \alpha z_c^* + (1-\alpha) z_a^* \end{cases}$$

for every $\epsilon_c \in (0, \frac{1}{n}]$ and $\epsilon_a \in (\frac{1}{n}, \frac{2}{n}]$.

- Fraction of agents playing $+1$ in $\mathcal{V}_c$ and $\mathcal{V}_a$

$$z_c^* := \frac{1}{n_c}\{i \in \mathcal{V}_c \mid x_i^* = +1\}, \qquad z_a^* := \frac{1}{n_a}\{i \in \mathcal{V}_a \mid x_i^* = +1\}$$

- Fraction of coordinating agents

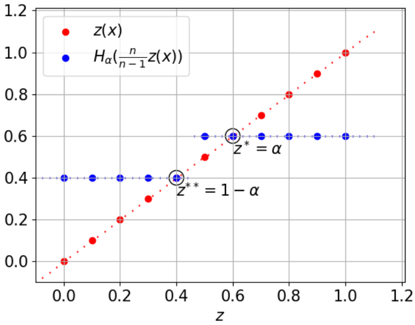
$$\alpha := \frac{n_c}{n}$$

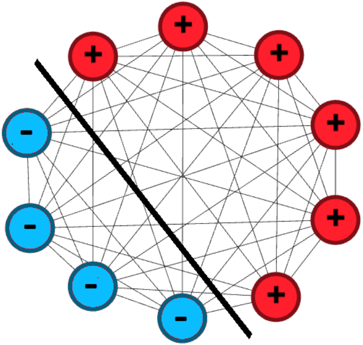
Nash equilibria on complete graph

Coexistence of coordinating and anti-coordinating agents (Necessary condition)

$$x^* \in \mathcal{N} \quad \Rightarrow \quad H_\alpha\left(\frac{n}{n-1}(z^* - \frac{1}{n})\right) \geq z^* \geq H_\alpha\left(\frac{n}{n-1}z^*\right)$$

where $H_\alpha(z) := \alpha F_c(z) + (1-\alpha)G_a(z)$





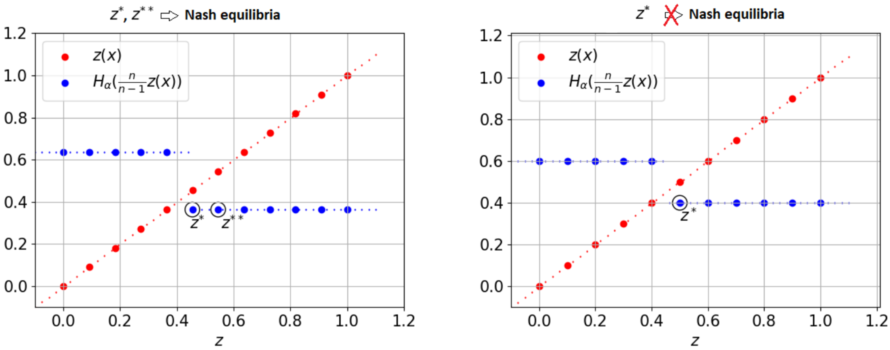
- $r_i = \frac{1}{2}$, $\forall i \in \mathcal{V}$: $\alpha > \frac{1}{2}$ $\rightarrow$ $\mathcal{V}_c$ is $\frac{1}{2}$-cohesive

Nash equilibria on complete graph

Coexistence of coordinating and anti-coordinating agents (Necessary condition)

$$x^* \in \mathcal{N} \quad \Rightarrow \quad H_\alpha\left(\frac{n}{n-1}\left(z^* - \frac{1}{n}\right)\right) \geq z^* \geq H_\alpha\left(\frac{n}{n-1} z^*\right)$$

where $H_\alpha(z) := \alpha F_c(z) + (1-\alpha) G_a(z)$



- $r_i = \frac{1}{2}, \forall i \in \mathcal{V}$: $\alpha \leq \frac{1}{2}$ $\rightarrow$ NE if n odd

Conclusions

Results

- We observed that the *heterogeneous network coordination game* and the *heterogeneous network anti-coordination game* are *potential games*
- Even if the potential property is formally lost, we provide a *sufficient condition* for the existence of *Nash equilibria* of the *heterogeneous network coordination anti-coordination game*
- Characterization of Nash equilibria of the heterogeneous network coordination anti-coordination game over the *complete graph*

Open questions

- The condition is sufficient but not necessary. *Necessary conditions?*
- We studied the static case. Let us consider the *asynchronous best response dynamics*. If the conditions of the theorem are satisfied, does the dynamics converge to a Nash equilibrium?

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Thank you for the attention

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