On imitation dynamics in potential population games

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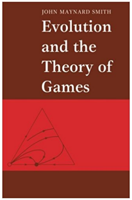
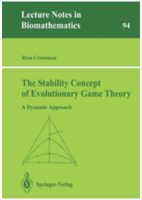
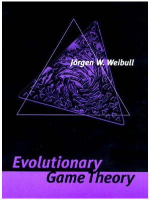
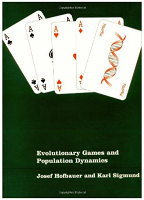
Learning and evolution in games

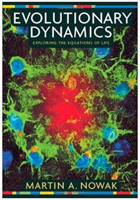
- Learning in Games
- Evolutionary Game Theory [Maynard Smith, Price, Cressman, Weibull, Sigmund, Hofbauer, Nowak, Sandholm,...]

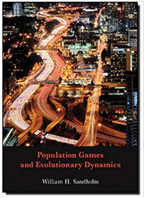












(Noisy) best response dynamics

- Players have **full information** on all the actions and the rewards
- They update their action choosing the one that **maximizes** the current reward (w/ or w/o noise)
- In the literature, many results estabish **convergence** to Nash and evolutionary stable states

(Noisy) best response dynamics

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not always realistic!

☹ In decision making, we have often **limited information**

☹ Information might be available but **hard to process** (e.g., big data)

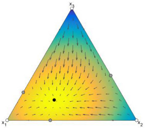
Imitation dynamics

- Players have **minimal information**: no knowledge of the game structure and action space, no memory
- Each player can measure its own current reward and **communicate** with fellow players its current action and reward
- Players can **update** their action using the information from the communication network

Outline

Introduce **potential population games** and notion of Nash equilibria and evolutionary stable states

Define the learning mechanism: **imitation dynamics**



Deterministic imitation dynamics: convergence to Nash equilibria of the population game

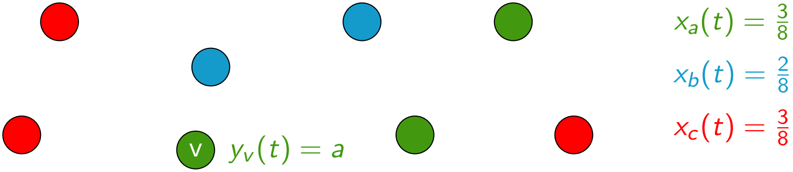
Stochastic imitation dynamics: new emerging behaviors, meta-stability of evolutionary stable states

Population games

- Population $\mathcal{V} = \{1, \dots, n\}$
- $\mathcal{A} = \{a, b, \dots\}$ finite set of **actions**
- $y_v(t) \in \mathcal{A}$: action played by player v at time $t \in \mathbb{R}$
- $x_a(t)$ **empirical frequency** of a-players at time t

$$x_a(t) = \frac{1}{n} \left| \{ y_v(t) = a \} \right|$$

- **Reward** $r_a(x)$ depends on empirical frequencies x (anonymous game)

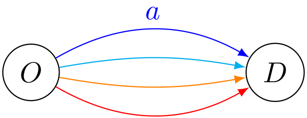


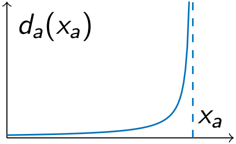
Potential population games

Population game, reward $r(x)$, is **potential** if $\exists\, \Phi : \mathcal{X} \to \mathbb{R}$

$$r_a(x) - r_b(x) = \frac{\partial}{\partial x_a}\Phi(x) - \frac{\partial}{\partial x_b}\Phi(x)$$

Example. Transportation network from **origin** O to **destination** D





- Action set $\mathcal{A} = \{\text{direct } O \to D \text{ paths}\}$
- x_a fraction of **drivers** on path $a \in \mathcal{A}$
- Reward $r_a(x_a) = -d_a(x_a)$, **delay** on path a (increasing in x_a)
- Potential $\Phi(x) = -\sum_{a\in\mathcal{A}} \Psi_a(x_a)$ (Φ_a anti-derivative of d_a)

Nash equilibria and evolutionary stable states

- Maximum reward vs average reward

$$r^*(x) = \max_{a \in \mathcal{A}} r_a(x), \qquad \bar{r}(x) = \sum_{a \in \mathcal{A}} x_a r_a(x)$$

- **Critical points** (of continuous game)

$$\mathcal{Z} = \{x \in \mathcal{X} : x_a > 0 \implies r_a(x) = \bar{r}(x)\}$$

- **Nash** equilibria $\mathcal{N} \subseteq \mathcal{Z}$ (of continuous game)

$$\mathcal{N} = \{x \in \mathcal{X} : x_a > 0 \implies r_a(x) = \bar{r}(x) = r^*(x)\}$$

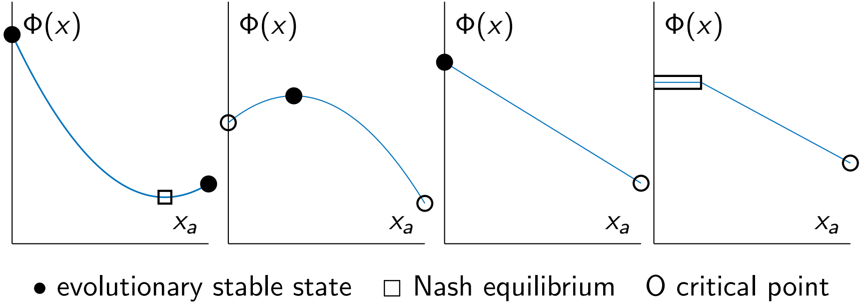
- **Evolutonary stable states** $\mathcal{S} \subseteq \mathcal{N}$ (of continuous game)

$$\mathcal{S} = \left\{x \in \mathcal{X} : \exists \epsilon > 0, 0 < ||x - y|| < \epsilon \implies (y - x)^T r(y) < 0\right\}$$

Nash equilibria in potential games

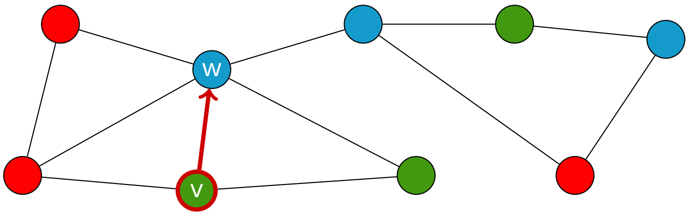
Folk theorems of evolutionary game theory [Sandholm 2010]

- $\bar{x} \in \mathcal{N} \iff \bar{x}$ stationary point of Φ
- $\bar{x}$ isolated local maximum of $\Phi \implies \bar{x} \in \mathcal{S}$
- All maxima of Φ isolated: $\bar{x} \in \mathcal{S} \iff \bar{x}$ maximum of Φ



Imitation dynamics

- Player v **contacts** (at random, over an undirected communication network) a fellow player w
- The **information** it can access: its own action a and reward $r_a(x)$; the action of the fellow player b and its reward $r_b(x)$
- It **updates** its action from a to b with probability $p_{ab}(x(t))$



Imitation dynamics (cont'd)

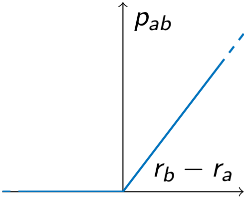
Assumption higher probability to update to increase reward

$$\text{sign}(p_{ab}(x) - p_{ba}(x)) = \text{sign}(r_b(x) - r_a(x))$$

- Example I: **proportional imitation** rule:

$$p_{ab}(x) = \alpha[r_b(x) - r_a(x)]_+, \quad \alpha > 0$$

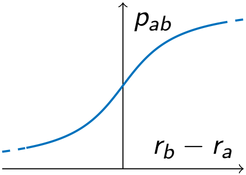
$\Longrightarrow$ **replicator** equation [Taylor & Jonker, 1978; Schuster & Sigmund, 1983]



- Example II: **nonlinear imitation** rule

$$p_{ab}(x) = \frac{1}{2} + \frac{1}{\pi}\text{atan}\left(K_{ab}(r_b(x) - r_a(x))\right)$$

$$K_{ab} = K_{ab}(x) > 0$$



Imitation dynamics for all-to-all communication

- Frequency of **pairwise interactions** of agents playing a and $b \propto x_a x_b$
- Overall rate of **transitions** from a to $b \propto x_a x_b p_{ab}(x)$
- For n large, finite time horizon, imitation dynamics $\approx$ **deterministic system** of ODEs [Kurtz, 1970]:

$$\dot{x}_a = x_a \sum_{b \in \mathcal{A}} x_b (p_{ba}(x) - p_{ab}(x)), \qquad a \in \mathcal{A}$$

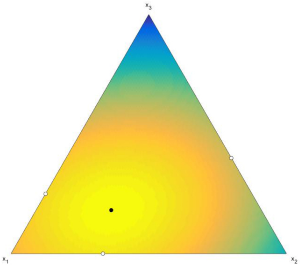
Convergence to Nash (LZ, G. Como, F. Fagnani, Proc. CDC 2017)

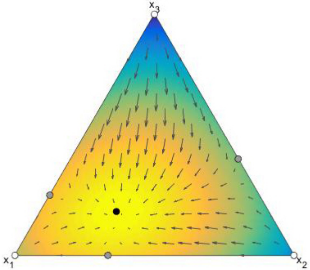
Deterministic imitation dynamics, potential population game, $x_a(0) > 0$, $\forall\, a \in \mathcal{A}$

$$x(t) \to \mathcal{N}$$

Sketch of the proof

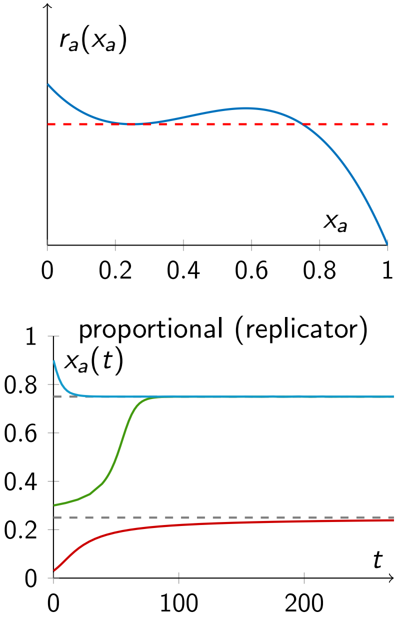
- ☑ **Potential** Φ cannot **cannot decrease** along trajectories
- ☑ **Stationary points** of $\Phi(x(t))$: ☺ Nash $\mathcal{N}$, ☹ $\mathcal{Z} \setminus \mathcal{N}$ critical points
- ☑ Every critical point $\bar{z} \in \mathcal{Z} \setminus \mathcal{N}$ is surrounded by an interior neighborhood with potential greater than $\Phi(\bar{z})$
- ☑ The dynamics must pass through this neighborhood before touching $\bar{z}$ (Gronwall's inequality) $\Longrightarrow$ we **exclude** converge to $\mathcal{Z} \setminus \mathcal{N}$



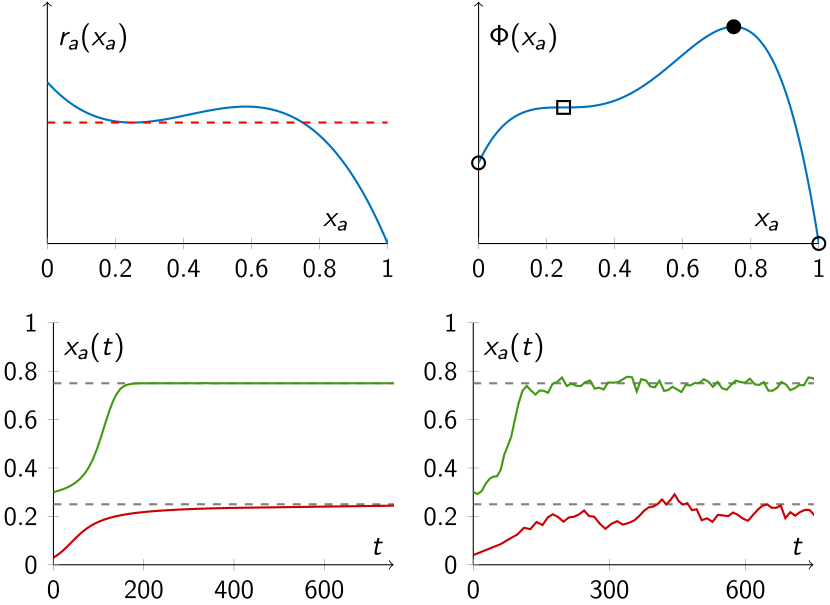


Remark If $x_a(0) = 0$, restricted games

Imitation dynamics (ODEs)...

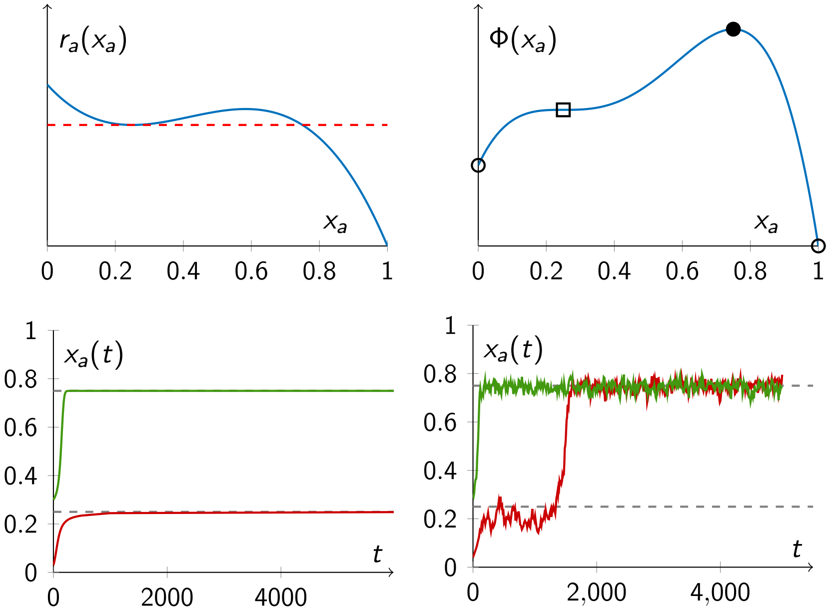


...vs. stochastic imitation



GAME OVER

...not at all!



$r_a(x_a)$
x_a
0 0.2 0.4 0.6 0.8 1
$\Phi(x_a)$
x_a
0 0.2 0.4 0.6 0.8 1
$x_a(t)$
1 0.8 0.6 0.4 0.2 0
t
0 2000 4000
$x_a(t)$
1 0.8 0.6 0.4 0.2 0
t
0 2,000 4,000

Stochastic imitation dynamics

Assumptions:

i) all-to-all communication
ii) full support ($x_a(0) > 0$)
iii) initial condition interior ($\sout{x_a(0)/n \to 0}$)
iv) finite number of critical points of Φ
v) if $|\mathcal{A}| \geq 3$ all critical points are maxima or minima (no saddle points)

Convergence to ESS (LZ, G. Como, F. Fagnani, Proc. ECC 2018)

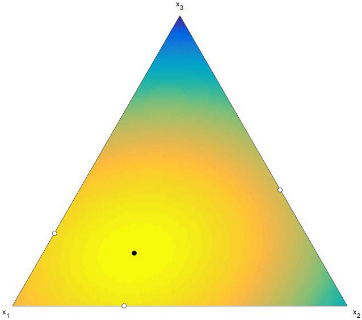
Potential population game, assumptions i)–v). Then, for any $\delta > 0$ there exist $C_1, C_2 \geq 0$ such that

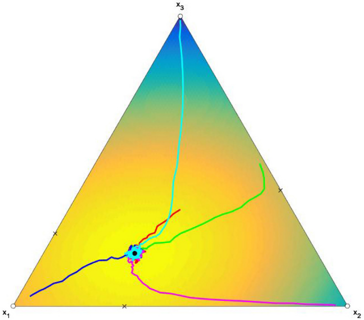
$$x(t) \in \mathcal{B}_\delta(\mathcal{S}), \qquad \forall t \in [C_1 n \ln n, e^{C_2 n}],$$

with high probability as $n \to \infty$, where $\mathcal{B}_\delta(\mathcal{S})$ is the δ-neighborhood of the set of evolutionary stable states.

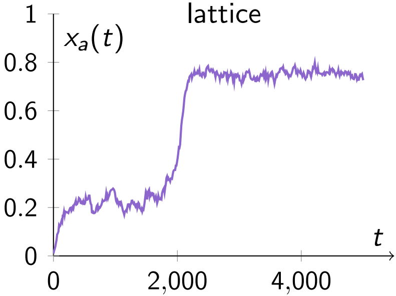
Sketch of the proof

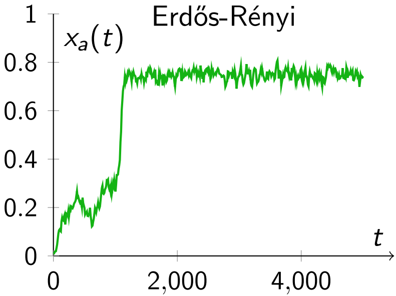
- ☑ Far from critical points, we use ODE [Kurtz, 1970] $\Longrightarrow$ imitation dynamics converges to the neighborhood of Nash (w.h.p.)
- ☑ In the neighborhood of ESS (maxima of the potential), an exponentially-long time is needed to decrease Φ (w.h.p.)
- ☑ In the neighborhood of Nash non ESS (minima of the potential), optional stopping Theorem yields exit time from the neighborhood in $kn \ln n$ (w.h.p.)

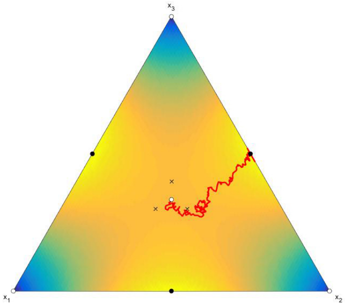


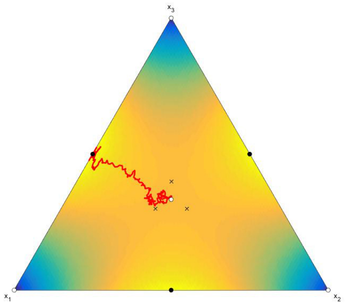


Generalization? Numerical simulations









Conclusions and future works

Imitation dynamics in potential population games

- ☺ Deterministic dynamics: convergence to **Nash**
- ☺ Stochastic dynamics: meta-stability of **evolutionary stable states**
- ☺ Simulations suggest **extension** (saddle points, non-all-to-all)

Current/future work

- Analysis of non-all-to-all communication
- Extend results for stochastic imitation (saddles, non isolated Nash)
- Beyond potential population games

More details can be found in...

- ✍ *On imitation dynamics in potential population games*, Proc. 56th Annual Conference on Decision and Control, pp. 757–762, 2017
- ✍ *On stochastic imitation dynamics in large-scale networks*, Proc. European Control Conference, pp. 2176–2181, 2018

Thank you for your attention!

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