Enumerative geometry after string theory

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Enumerative geometry **before** string theory

- Counting points
- Counting curves
- Compactifying moduli
- Local vs global expected dimension

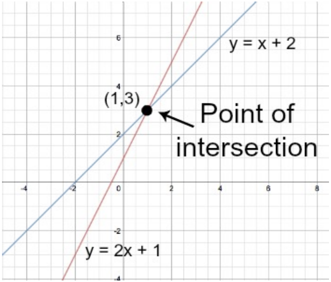
String theory!

- First interaction
- Enter Witten
- Making derivations into proofs
- Symplectic geometry
- Algebraic geometry
- Obstruction theories and virtual classes

The next quarter century

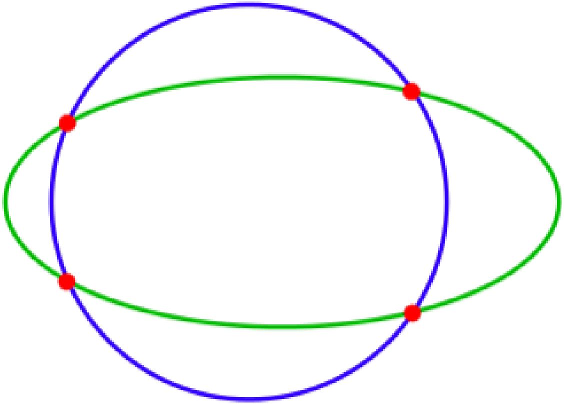
Counting points

1. Two lines in a plane meet in **one** point.
2. Bezout's theorem: a curve of degree d and a curve of degree e meet in de points.



Bezout's theorem

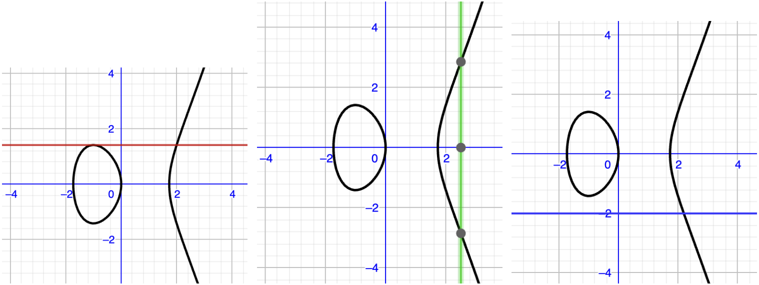
Given two plane curves C_1 of degree d_1 and C_2 of degree d_2, they have **exactly** $d_1 d_2$ points in common.



Well, "exactly"...

- If the two curves are tangent at the intersection point, we need to use **multiplicities**.
- Points can "escape" at infinity. We use **projective** geometry.
- Some points only exist if one allows complex coordinates. We fix an **algebraically closed** base field ($\mathbb{C}$ in this talk).

And "of course" we have to exclude the case where there is a common subcurve (this will come back!)



Counting lines

- There is **one** line through two given points.
- Given four lines in space, there are **two** lines intersecting all of them.
- Given a degree 3 polynomial $f(x, y, z)$ (cubic surfaces), there are **27 lines** contained in the surface $f = 0$.
  In each case we assume the input data to be "generic" (outside a set of measure/probability zero). In general, we count curves, or anything else, only if the "expected dimension" is zero.

Calculate like an algebraic geometer!

Consider the curve C of equation $\{y + x^3 = 0\}$ in the plane $\{z = 1\}$, and let X be the cone over C with vertex in the origin.

The equation of X is $f(x, y, x) := yz^2 + x^3 = 0$.

What is the space of lines contained in X? Let's work near the line $(0, 0, t)$.

We can parametrize the line as $(a + tu, b + tv, t)$ where (a, b, u, v) are parameters determining the line, and t is a coordinate on the line. We impose the condition $f(a + tu, b + tv, t) = 0$ as polynomial of degree 3 in t; we get four equations.

- $v + u^3 = 0$;
- $b + 3au^2 = 0$;
- $3a^2u = 0$;
- $a^3 = 0$.

We can solve in v and b and get two free parameters a, u with conditions $3a^2u = a^3 = 0$. This does NOT mean $a = 0$!

Counting curves of given degree d and genus g

Some more classical results in enumerative geometry of the plane:

1. There is **one** degree two, genus zero through 5 given points;
2. There is **one** degree three, genus one curve through 9 given points;
3. There are **twelve** degree three, genus zero curves through 8 given points.

Open question (before string theory): How many plane curves of degree d and genus g are there through $N(d,g)$ points? There is an explicit formula for $N(d,g)$, which is chosen so that the expected dimension is zero. NB Here the genus is the geometric genus, ie the genus of the normalization.

Gromov's compactness theorem

Just as in the counting point case, the first step is to compactify the space of curves.
In 1985, M. Gromov constructed a **compact** moduli space for maps from complex curves to symplectic manifolds (in particular all smooth complex projective varieties/projective manifolds).
Gromov's construction got a counterpart in algebraic geometry due to Kontsevich and Manin.

(a) Mikhail Gromov

(b) Maxim Kontsevich

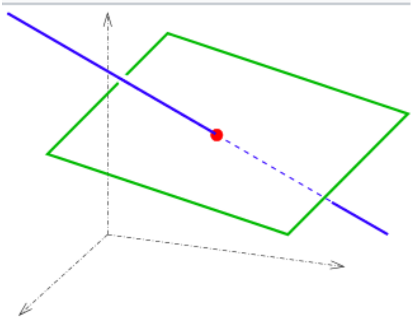
(c) Yuri Manin

Unfortunately the moduli space so constructed isn't a manifold. It doesn't have local coordinates, nor a dimension.

All it has is an "expected dimension" D: ie, locally it can be naturally described by N equations in $N + D$ variables. However different points will in general have different N (and in algebraic geometry, local means étale not Zariski local). The actual dimension may vary.

(Whispering: technically, it isn't even a space but a Deligne Mumford stack).

Example: two equations in three variables give expected dimension 1, but $xy = xz = 0$ is the union of a line and a plane.



First interaction

In 1991, physicists Philip Candelas, Xenia De La Ossa, Paul Green, and Linda Parkes *derived* a power series formula for all numbers of rational curves on a quintic threefold.
This agreed with the known cases $d = 1, 2$.

Soon after, mathematicians Geir Ellingsrud and Stein Arild Strømme calculated rigorously the case $d = 3$... and got a different number!

The mathematicians had made a programming mistake, and when it was fixed, their number agreed with the physicists' prediction.

Witten: applying physics to maths

Based on string theory ideas, Edward Witten asked himself in 1991: what if we **pretend** the Gromov moduli space is a manifold of the expected dimension?

He then showed that, if this were the case, a lot of counting curves problems could in principle be solved: the solutions have been called **Gromov-Witten invariants**.

He combined the invariants in a partition function Z, and he and other physicist "derived" that Z must satisfy a series of partial differential equations, PDEs. In some cases such PDEs are enough to compute the invariants.

Interesting, promising results... based on an assumption that was *known to be wrong*.
Kontsevich and Manin produced an axiomatization of Witten's argument. That is, they made explicit the numerical assumptions he had used, and posed the problem of how to **define** Gromov-Witten invariants in general so that they would do what Witten had proposed they should do.
Kontsevich then showed that, if KM axioms could be made rigorous, the problem of counting plane curves of genus zero and any degree could be solved recursively.
Goal for mathematicians: replace "derivation" by PROOF. The hunt begins!

Algebraic vs. symplectic geometry

There were two overlapping communities trying to solve the problem.

Symplectic geometry is a variation of differential geometry. It is strongly related to mathematical physics, as the easiest example of symplectic manifold is the phase space of a system of particles, recording for each of them position and speed. Its methods are analytic, extension of calculus in several variables.

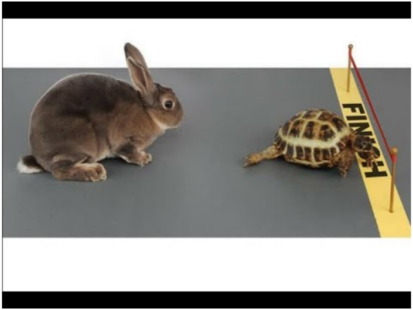
Algebraic geometry is the study of shapes that can be described not by arbitrary equations, but by polynomials. It is narrower in scope than symplectic geometry, and technically more demanding. It does, however, have a much easier grasp of so-called singularities.

So... who won the race?

Symplectic geometry approach

Symplectic geometry led the race for a long time. Invariants with the desired properties were defined, first in special cases, then in big generality, then in increased generality. But the full generalization didn't come.

In the meantime, algebraic geometers were making slow progress.
Spoiler



Algebraic geometry approach

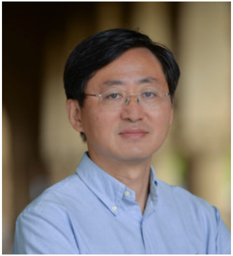
Kai Behrend and Yuri Manin recast the Kontsevich Manin axioms not in terms of numbers, but of a **virtual class**, a homology class on the moduli space of the expected dimension, over which to make integrals.

This wasn't a new idea in algebraic geometry, and in fact a very general formulation had been produced by Fulton and MacPherson; however such class existed only if there was a global presentation, not only a local one.

Race completed!

In 1996, Jun Li and Gang Tian gave a construction of the virtual class, in full generality. This required producing a global object over the moduli space (technically, a pure dimensional cone), embed it into a vector bundle, and then apply the Fulton-MacPherson method.

The key input is a so called tangent-obstruction complex, and the technique involves local choices, gluing, and showing that the result doesn't depend on the choices.

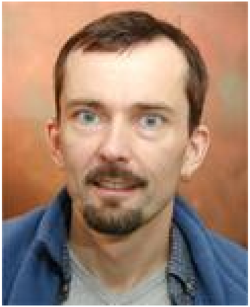


(a) Jun Li

(b) Gang Tian

Why am I telling you this?

In 1997, inspired by Li and Tian's work, Behrend and I were able to construct a virtual class under very general assumptions. In particular, we showed that Li and Tian's cone, which they constructed by patching together local charts, was in fact induced by a canonical object, the intrinsic normal cone (which in turn could be "found" already in Fulton and MacPherson's construction).




Ingredients in our recipe

Our construction follows closely the method of Fulton MacPherson's intersection theory. However, we replace cones and vector bundles by their stack counterparts; we also use the ideas in Deligne's work on Picard stacks to relate vector bundle stacks on X to the derived category of X.

In analogy with Li and Tian, the input is a complex in the derived category with a map to the cotangent complex. The necessary condition, namely that we get a closed embedding from the normal cone into an ambient bundle, can be recast into infinitesimal deformation theory, for which there are ample references.

Gromov Witten invariants: computational techniques

Tom Graber and Rahul Pandharipande used our construction to prove a virtual version of the classical localization formula in intersection theory, which became one of the key computational methods in the field.

The construction led naturally to proving relationships between enumerative invariants, without actually computing them. For instance, Kontsevich's original formula counts recursively the number N_d of curves of genus zero and degree d through $3d - 1$ generic points in the plane, using as seed just the number N_1.

Gromov Witten invariants are related to Frobenius manifolds, which brought fruitful interactions with the research in integrable systems.

Donaldson invariants

The original motivation for Li and Tian had been not just to give a definition of Gromov Witten invariants, but also to redefine Donaldson invariants, heavily based on analysis methods, in a purely algebraic geometry set-up.
This became indeed possible, but not in full generality because of a technical issue with strictly semistable sheaves. (This has since been fixed but adapting to algebraic geometry a recent technique by Frances Kirwan in GIT theory).
So, in this case the analytical methods were better? Well...

Donaldson-Thomas invariants

In 1998, Simon Donaldson proposed to his PhD student Richard Thomas to construct analogues of Donaldson invariants for 6-dimensional symplectic manifolds.

After trying analytic/symplectic methods, Thomas finally gave a construction using the algebraic virtual class (on a 3-dimensional complex projective manifold Y), under the assumption that K_Y is trivial or $-K_Y$ effective.

In particular, the case where $K_Y = 0$, so called Calabi Yau threefolds, was interesting for both geometers and physicists.

(a) Simon Donaldson (b) Richard Thomas

What next? Outside algebraic geometry

- Theoretical physics
- Integrable systems
- Symplectic geometry
- PDEs on formal power series
- Index of Fredholm operators among Banach orbibundles

What next? Gromov Witten

- Relative GW invariants, degeneration formulas
- GW for orbifolds
- open GW invariants
- GW invariants in characteristic p
- Other invariants (non-contracting maps, quasimaps...)

What next? Sheaf counting

- Invariants in CY threefolds, including noncompact ones
- Wall crossing methods
- Invariants for Quot schemes
- Degeneration formulas
- Behrend function

Theoretical advances, mild

- Virtual pushforward and pullback
- Virtual structure sheaf
- Symmetric obstruction theories
- Use of master spaces
- Virtual Grothendieck Riemann Roch

Infinity category advances

- Quasismooth derived schemes
- Shifted symplectic structures
- Categorification of Behrend function
- Trimester at IHP 2023!

Thank you!