Stability and Recursive solutions in Hamiltonian PDEs

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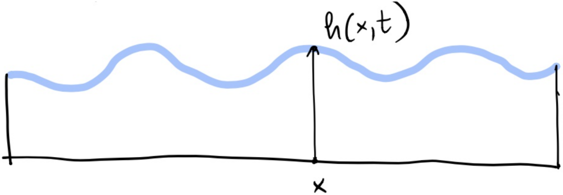
Non linear PDE's are ubiquitous in Mathematics and Physics. Some of the most famous examples come from Hydrodynamics: such as the Korteweg de Vries, Non Linear Schrödinger, Camassa Holm etc.

$$(\text{NLS}) \qquad -\mathrm{i}u_t + u_{xx} + |u|^2 u = 0\,,$$

$$(\text{KdV}) \qquad h_t - h_{xxx} + hh_x = 0\,,$$

$$(\text{CH}) \qquad h_t - h_{xxt} - 4h_x + hh_{xxx} + 3hh_x - 2h_x h_{xx} = 0$$

all these equations model one dimensional waves.. of course physical models are in dim. three , but one looks for plane waves!



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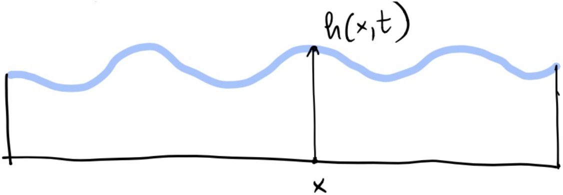

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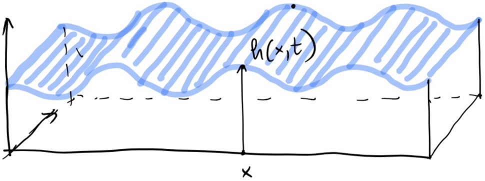
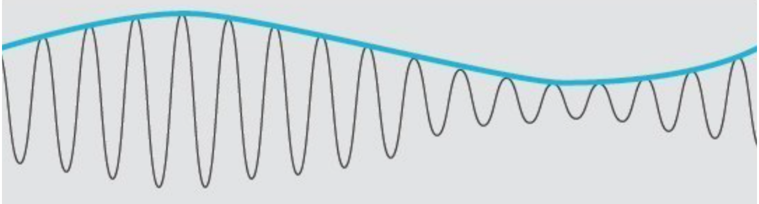


The NLS appears naturally as a "modulation equation"



which means you look for a wave profile

$$h(x,t) = \mathrm{Re}(u(\varepsilon x, \varepsilon^2 t)e^{\mathrm{i}(x+t)}) + \text{corrections}$$

where (for example)

$$\text{(NLS)} \qquad -\mathrm{i}u_\tau + u_{yy} + |u|^2 u = 0\,,$$

One might be interested also in modeling waves in higher dimensional domains (Δ is the Laplacian)

$$\begin{cases} -i\partial_t u + \Delta u = |u|^{2p} u, \\ u = u(t,x)\,, \quad x \in D \end{cases} \qquad (NLS)$$

I could be more refined and work on a Riemanian manifold .

I shall mostly concentrate on the NLS model on very simple compact domains where I expect to see persistent wave phenomena.
Consider an initial value problem

The Cauchy problem:

$$\begin{cases} -i\partial_t u + u_{xx} = |u|^2 u, \\ u(0,x) = u_0(x)\,, \quad x \in [0, 2\pi] \end{cases} \qquad (NLS)$$

where $u_0(x)$ is a sufficiently smooth periodic function.

what can I say about the solution?

$$\text{(KdV)} \qquad h_t - h_{xxx} + hh_x = 0\,,$$

$$\text{(CH)} \qquad h_t - h_{xxt} - 4h_x + hh_{xxx} + 3hh_x - 2h_x h_{xx} = 0$$

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actually all the 1D equations above are completely integrable so I could "explicitly determine" the solution!

However significant results should be **Robust**:

i.e. hold also if I make a small perturbation or if I slightly change the initial datum

Remember that these PDEs are just approximate models!

Over very short time scales I can ignore perturbations

but I am interested in long time behavior!

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This is a well known framework for finite dimensional Hamiltonian systems,
where you have an integrable model (you know everything about the solutions)
and you study the dynamics of small perturbations of this system.

Let us Forget the integrability and start by looking for small solutions so that (at least for finite times) we can treat the non-linearity as a perturbation

$$\text{(KdV)} \qquad h_t - h_{xxx} + hh_x + \text{h.o.t} = 0\,,$$

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if I rescale $h \rightsquigarrow \varepsilon h$, this just amounts to adding an ε in front of the non-linear term

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$$(\text{NLS}) \qquad -\mathrm{i}u_\tau + u_{yy} + \varepsilon^2(|u|^2 u + \text{h.o.t}) = 0\,,$$

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Natural questions

If I start with a small initial wave profile (in some function space)

- Is there a "typical behavior" of solutions?
- If so, which is it and for how long does it persist?

Natural questions

In all the examples I made I know all solutions of the linear PDE.
Questions:

- For which time scales do the all solutions of the nonlinear equation stay close to corresponding solutions of the linear one? (a trivial estimate is $T \ll 1/\varepsilon$)
- Are there nonlinear solutions which stay close to linear ones for all times?
- What new phenomena appear due to the presence of the non-linearity?

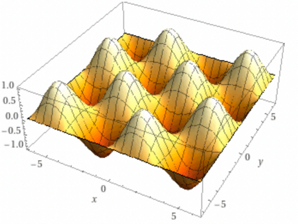
Remember that the answers should depend strongly on the boundary conditions.

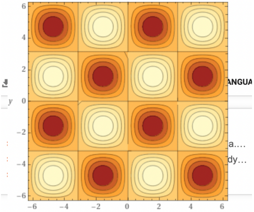
NonLinearSchrödinger on tori

I will concentrate on NLS equations on $\mathbb{T}$ and $\mathbb{T}^2 = \mathbb{R}^2 \setminus 2\pi\mathbb{Z}^2$

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u, \\ u = u(t,x)\,, \quad x \in \mathbb{T}^2 \end{cases} \qquad (NLS)$$

because here I have an interesting dynamics and a very natural occurrence of recursive waves in a relatively simple model.





In this framework

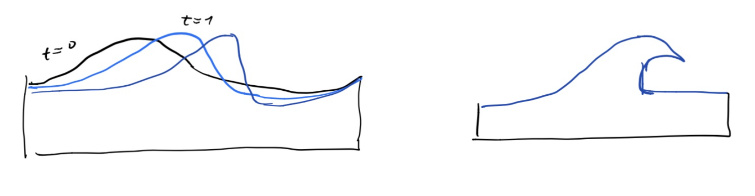
The Cauchy problem:

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u, \\ u(0,x) = u_0(x) \quad x \in \mathbb{T}^2 \end{cases} \qquad (NLS)$$

where $u_0(x)$ is a sufficiently smooth function.

This problem is globally well posed for u_0 small.
If you start with a smooth initial datum the solutions stays smooth at all times.

The NLS does not model wave-breaking



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The 2D NLS is not integrable
we cannot expect to be able to solve explicitly...and solutions should depend strongly on initial data.

look for small solutions
just rescale $u \to (\varepsilon)^{1/2} u$

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Linear theory

First we study the solutions at $\varepsilon = 0$

$$\begin{cases} -i\partial_t u + \Delta u = 0, \\ u(0,x) = u_0(x)\,, \quad x \in \mathbb{T}^2 \end{cases}$$

It is useful to describe the solution in the Fourier modes:

$$u(0,x) = \sum_{k \in \mathbb{Z}^2} u_k(0) e^{\mathrm{i}k\cdot x} \quad \Longrightarrow \quad u(t,x) = \sum_{k \in \mathbb{Z}^2} u_k(0) e^{\mathrm{i}k\cdot x + \mathrm{i}|k|^2 t}$$

Linear theory: $-i\partial_t u + \Delta u = 0$

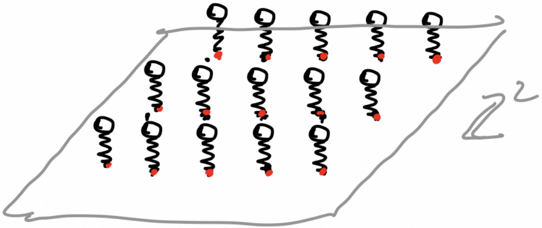
each Fourier mode oscillates independently with frequency $\lambda_k = |k|^2$.

$$\dot{u}_k(t) = \mathrm{i}|k|^2 u_k\,, \qquad \Longrightarrow \qquad u_k(t) = u_k(0)e^{\mathrm{i}|k|^2 t}$$

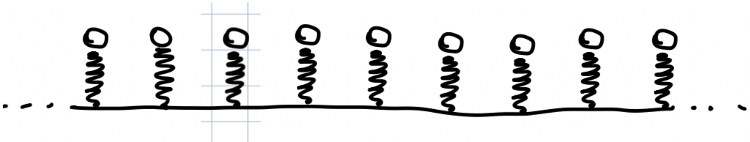
ALL solutions of the linear Schrödinger equation are PERIODIC of period 2π

Possible interesting phoenomena

At $\varepsilon = 0$, we have an infinite chain of independent springs



to simplify ...think of 1d



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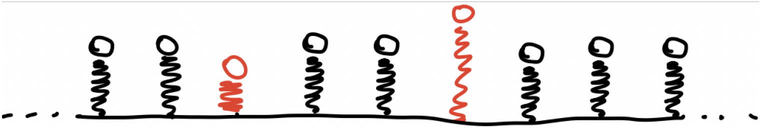
At $\varepsilon = 0$, we have an infinite chain of independent springs If we give energy to a finite number...



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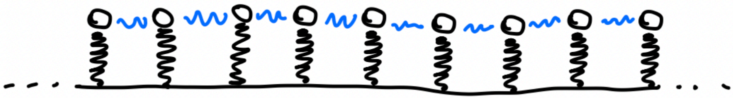

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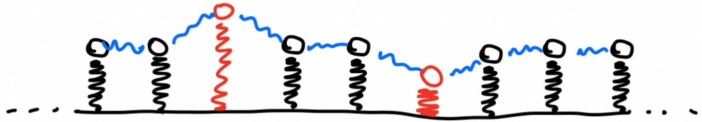

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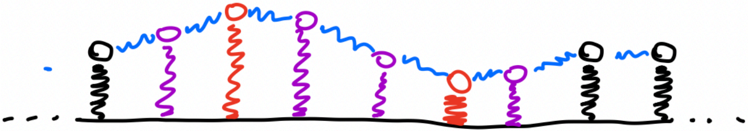
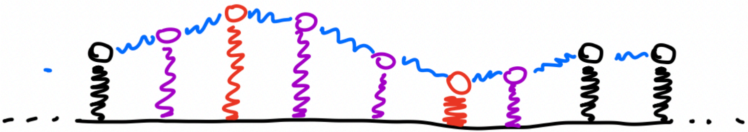


Possible interesting phoenomena

Actually the model



is a strong simplification nearest neighbor interaction! In the NLS

$$\dot{u}_k = \mathrm{i}|k|^2 u_k + \varepsilon \sum_{k_1+k_2-k_3=k} u_{k_1} u_{k_2} \bar{u}_{k_3}.$$

all points forming a parallelogram $k_1 + k_2 - k_3 - k = 0$ have an interaction!

The stability question.

A very reasonable question is to study the time evolution of the Sobolev norms

$$|u|_s^2 := \sum_{k \in \mathbb{Z}^2} |u_k(t)|^2 (1 + |k|^{2s})$$

If $|u(t=0)|_s < \infty$ then $|u(t)|_s < \infty$... can I control the time evolution of $|u(t)|_s$?

The smallness in Sobolev norm means $|\widehat{u}_k| \sim |k|^{-s}$ as $k \to \infty$

so the bigger is s the more is u localized on the low modes.

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Assume that $u_0(x)$ is small in some appropriate Sobolev norm, I want to know for how long the solution $u(x,t)$ is small in the same Sobolev norm.

Stability:

$$\text{If } |u_0(x)|_s \le \delta\,, \quad u(x,t) \text{ exists and satisfies } |u(\cdot,x)|_s \le 2\delta$$

for all $|t| < T(s,\delta)$.

the main point is to get a good lower bound on $T(s,\delta)$.

The stability question.

It is useful to describe the solution in the Fourier modes:

$$u(t,x) = \sum_{k \in \mathbb{Z}^2} u_k(t) e^{\mathrm{i} k \cdot x}$$

if we ignore the nonlinearity then $|u_k(t)|^2$ is constant in time.
If the Sobolev norm grows then we are transferring energy from low modes to high modes

Possible interesting phoenomena: special solutions

I can also look for particular global solutions

1. Recurrent behavior

2. Energy transfer

Possible interesting phoenomena: special solutions

I can also look for particular global solutions

1. Recurrent behavior

Start from an initial datum which is essentially localized on a finite number of Fourier modes...
the solution stays essentially localized on the same modes at all times.

2. Energy transfer

Possible interesting phoenomena: special solutions

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1. Recurrent behavior

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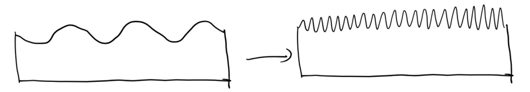
Start from an initial datum which is essentially localized on a finite number of Fourier modes...
the Fourier support of the solution spreads to higher modes.



Dynamical systems in ∞ dimension.

we consider a PDE on a compact manifold as a non-linear dynamical system:

$$\dot{u} = F(u), \qquad u \in V$$

where

- V is a vector space (in our case a scale of Hilbert spaces)
- F is a non-linear functional from V in itself.

PDE examples.

All the PDEs in my examples “fit this setting”, for example

$$-i\partial_t u + \Delta u = |u|^2 u, \quad , \quad x \in \mathbb{T}^n$$

recalling

$$V \equiv \cup_{p\in\mathbb{R}} H_p := \{u(x) = \sum_{j\in\mathbb{Z}^n} u_j e^{\mathrm{i} j\cdot x}\,, \quad \sum_j \langle j\rangle^{2p} |u_j|^2 < \infty\}$$

$(\langle j\rangle := \max(1, |j|))$
we get

$$\dot u_j = \mathrm{i}|j|^2 u_j + \mathrm{i} \sum_{j_1 - j_2 + j_3 = j} u_{j_1} \bar u_{j_2} u_{j_3}$$

If we consider $u_t - u_{xxt} - u_x + u_{xxx} + uu_{xxx} + 3uu_x - 2u_x u_{xx} = 0$
we get

$$\dot u_j = \mathrm{i}\lambda_j u_j + P_j\,, \quad \lambda_j \in \mathbb{R}$$

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$$\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j\,, \quad \lambda_j \in \mathbb{R}$$

Dynamical systems in ∞ dimension.

I shall study my system close to an elliptic fixed point that is

$$\dot{u} = F(u) = Lu + P(u)$$

where L is a (typically unbounded) linear operator with

pure point purely immaginary spectrum

while P is a non-linear term (which has a zero of degree at least two at $u = 0$).
Thus one can reduce to a system of the form

$$\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j(u)\,, \quad u = \{u_j\}_{j\in I}$$

where $\lambda_j \in \mathbb{R}$. This is a chain of harmonic oscillators coupled by a non-linearity.

Linear theory

If take u small, $P(u)$ is "perturbative" w.r.t. the linear terms...

$$\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j(u)$$

The unperturbed system is a chain of uncoupled harmonic oscillators with frequencies λ_j.

$$\dot{u}_j = \mathrm{i}\lambda_j u_j \Rightarrow u_j(t) = e^{\mathrm{i}\lambda_j t} u_j(0)$$

So the linear solutions are a linear superposition of oscillating motions
If the λ_j are all rationally dependent (say all integers) then the solution is periodic in time.
Otherwise if the frequencies are rationally independent it is called a quasi-periodic motion.

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Quasi-periodic solutions and invariant tori

Recalling that each $u_j(t) \in \mathbb{C}$ spans a circle
All the solutions are dense on invariant tori with dimension depending on the support of the solution

$$\mathcal{S} := \{j \in \mathbb{Z}^n : \quad u_j(0) \neq 0\}$$

and on whether the λ_i are rationally independent.

$$u(t,x) = \sum_j u_j(0) e^{\mathrm{i}\lambda_j t + \mathrm{i} j \cdot x} = \sum_{j \in \mathcal{S}} \sqrt{\xi_j} e^{\mathrm{i}\lambda_j t + \mathrm{i} j \cdot x}$$

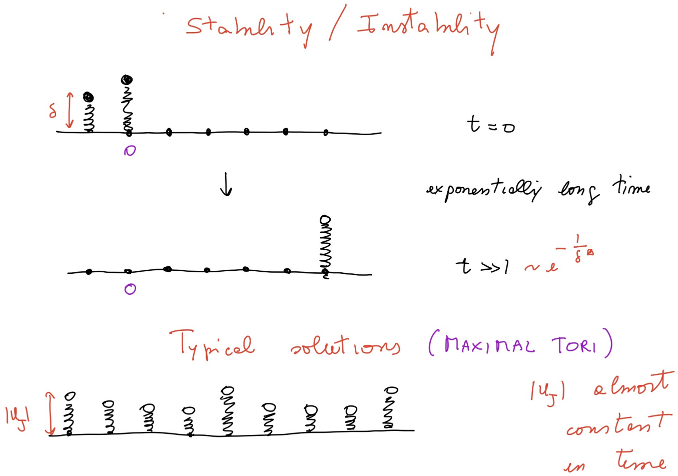
which solutions of the linear system survive the onset of the non-linearity?

finite dim. $\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j$

In finite-dimensional nearly integrable Hamiltonian systems,
Under some non-degeneracy assumptions (on λ or on P)

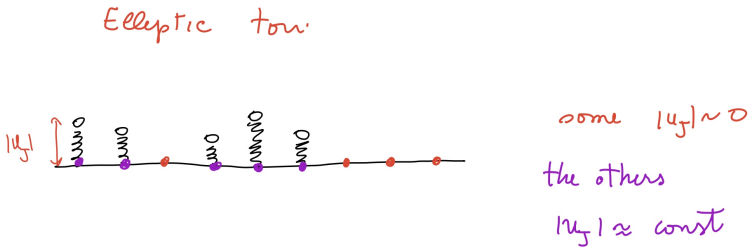
- Exponential stability: the actions $|u_j|^2$ are approx. constant for exponentially long times
- Typical recurrent behavior: the majority of small initial data give rise to quasi-periodic solutions. with diophantine frequency
- The KAM theorems predict persistence of most but not all quasi–periodic orbits. In the complementary set to the quasi–periodic orbits one may see chaotic behavior .

Summarizing: In finite dimension for most λ, P



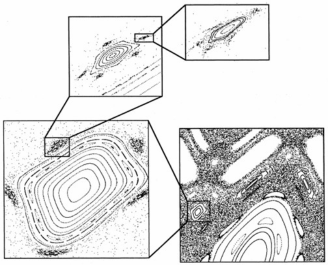
Summarizing: In finite dimension for most λ, P

One can also find lower dimensional tori



The appearance of Cantor sets

- The KAM theorems predict persistence of most but not all quasi–periodic orbits, parametrized by Cantor sets, at the same time one has the existence of infinitely many orbits, near each quasi–periodic orbit, which present cahotic behaviour.



- The complications arise from *small divisors* and *resonances* so that the success of the algorithm depends on suitable *non degeneracy conditions* which must be treated by a mixture of combinatorial and analytic methods.

What happens in infinite dimension?

Stability for PDEs

There is a vast literature on stability of elliptic fixed points!

Stability:

$$\text{If } |u_0(x)|_s \leq \delta\,, \quad u(x,t) \text{ exists and satisfies } |u(\cdot,x)|_s \leq 2\delta$$

for all $|t| < T(s,\delta)$.

Stability: Bourgain, Bambusi, Delort, Faou,Grebert, Szeftel, Yuan-Zhang, Cong-Mi-Wang, many more ....
The main point is to remember that the stability times depend very strongly on the choice of the function space!
Also in PDEs with derivatives in the nonlinearity also the question of local well posedness is delicate!

Some results for instability

To see instability results we need to consider NLS on $\mathbb{T}^2$!

- Breakthrough results by
  Colliander-Keel-Staffilani-Takaoka-Tao 2010: growth (of a finite but arbitrarily large factor) of Sobolev norms for the two-dimensional cubic NLS

  Kaloshin-Guardia 2013: growth of Sobolev norms for the cubic NLS with control on the time.

  Haus-P for the quintic NLS and Haus-Guardia-P

  Related work : Giuliani-Guardia-Pasquali-Pau

Typical solutions in ∞-Dim?

In the finite dimensional setting quasi-periodic solutions are "typical"

- In ∞-dim the scene is so far rather obscure:
  in an integrable PDE, almost-periodic solutions are typical and lie on maximal infinite dimensional invariant tori.
  what is their fate after perturbation? Is it still true that the majority of initial data produces perpetually stable solutions?
- There is a wide literature for existence of quasi-periodic solutions mostly for semilinear PDEs
  such solutions are NOT typical and correspond to lower dimensional tori
- there are very few results on infinite dimensional tori mostly for not very natural models.
- There are examples of PDEs which exhibit diffusive orbits.

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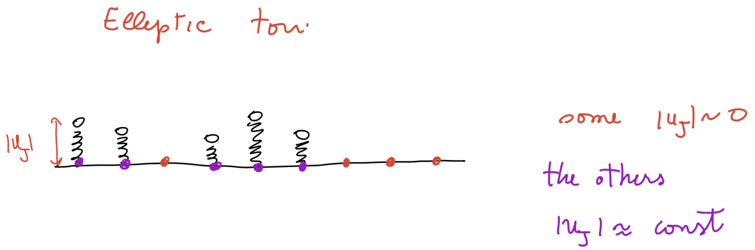
Summarizing: In infinite dimension

Many results on stability,
partial results on instability
No information on typical solutions. Most results on elliptic tori (KAM for PDEs)



KAM for PDEs

The first results were on model Hamiltonian PDEs such as the semilinear NLS with Dirichlet boundary conditions

$$-\mathrm{i}u_t + u_{xx} + |u|^2 u + g(x,u) = 0, \quad u(t,0) = u(t,\pi) = 0$$

- *(Semilinear PDEs with Dirichlet b.c.* : Kuksin, Wayne, Pöschel, Kuksin-Pöschel, Periodic b.c. Chierchia-You , Craig-Wayne '93 (periodic solutions), Bourgain '94 (quasi periodic solutions),.
- Higher dimensional manifolds:
  Tori: Bourgain '98,'05, Wang '10-'15, Berti-Bolle '10- '15, Geng-You, Eliasson-Kuksin '10, Geng-You-Xu '10, Procesi-P. '11-'15,
  Compact Lie groups Berti-Corsi-P., Grébert-Paturel '16

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Some more literature: unbounded non linearities

- **semi-linear Pde's**
  Kuksin '98, Kappeler-Pöeschel '03 KdV, Liu-Yuan '10, Zhang-Gao-Yuan '11 Hamiltonian and Reversible DNLS Berti, Biasco, P. , Hamiltonian and Reversible DNLW
- **Quasi-linear or fully non-linear Pde's**
  periodic solutions: Ioss-Plotnikov-Toland '01- '10, Alazard,Baldi capillary water waves
  quasi-periodic solutions: Baldi, Berti, Montalto '11 Airy,Baldi, Berti, Haus, Montalto, Feola-Giuliani Berti-Kappeler-Montalto
  Higher Dimension: Corsi-Montalto, Baldi-Montalto, Feola-Grebert

$iu_t - \Delta u + |u|^{2p} u = 0$ with $x \in \mathbb{T}^2$

quasi periodic solutions

Theorem (Procesi-M.P. 15)

For any $d \geq 1$. For most choices of Fourier modes $S = \{\mathtt{j}_1, \dots, \mathtt{j}_d\} \subset \mathbb{Z}^2$, one has that for many $\xi = \xi_1, \dots, \xi_d$, there exists a quasi-periodic solution of NLS of the form

$$u(\xi, x, t) = \sqrt{\varepsilon}\Big(\sum_{i=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}t(|\mathtt{j}_i|^2 + \omega_i(\xi))} e^{\mathrm{i}\mathtt{j}_i \cdot x} + O(\varepsilon)\Big)$$

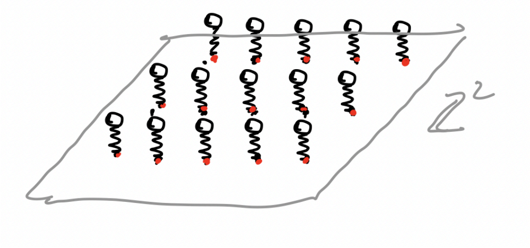
For many choices of $\xi_1, \dots, \xi_4$
there is a true solution close to the linear solution

$$\sqrt{\varepsilon}(\sum_{i=1}^{4} \sqrt{\xi_i} e^{\mathrm{i}t(|\mathrm{j}_i|^2)} e^{\mathrm{i}\mathrm{j}_i \cdot x}$$

the true solution looks like

$$u(\xi, x, t) = \sqrt{\varepsilon}(\sum_{i=1}^{4} \sqrt{\xi_i} e^{\mathrm{i}t(|\mathrm{j}_i|^2 + \omega_i(\xi))} e^{\mathrm{i}\mathrm{j}_i \cdot x} + O(\varepsilon))$$

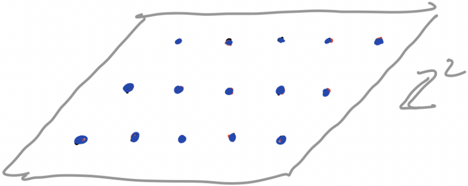
In our picture with springs



(blue dots are at rest)

give energy to modes in S.

In our picture with springs



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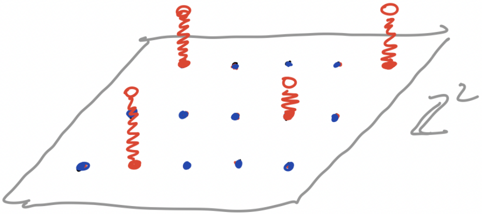
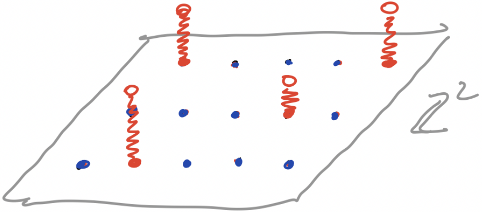


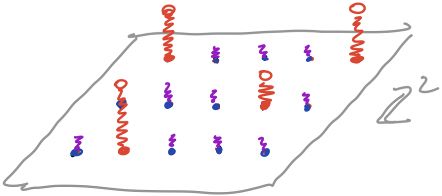
Linear solution

In our picture with springs (blue dots are at rest)

give energy to modes in S.



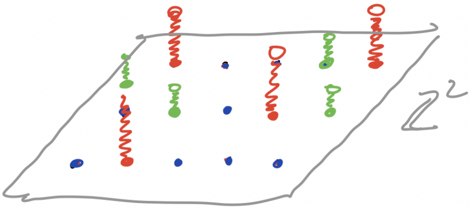
Linear solution



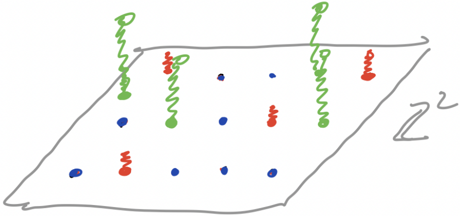
True solution

NB. near to these solutions we expect INSTABILITY!

If we take special values of the actions of the springs in S and give a little energy to the springs in S_2 (in green) Time zero



time $T \gg 1$



Summarizing

Some comments:

- Quasi-periodic solutions are very special, but their persistence is quite general (a similar approach can be used for other PDEs or for the NLS on other compact domains)
- The result for the instability is very model depending. Already passing from the cubic NLS to the quintic requires some very new strategies.

Drawbacks

- Quasi-periodic solutions are NOT typical even for integrable equations.
- The quasi-periodic solutions described are all at least C^∞ (both in time and space)
- In order to find finite regularity solutions one needs a non-linearity with finite regularity, and even in this case the regularity is very high.

To overcome these difficulties it is natural to look at almost-periodic solutions.
Very few results, most on 1d NLS or NLW with external parameters.

$$\mathrm{i}u_t - u_{xx} + V \star u + |u|^4 u = 0\,, \quad V \star u = \sum_{j\in\mathbb{Z}} V_j u_j e^{\mathrm{i}jx}$$

$\mathrm{i}u_t - u_{xx} + V \star u + |u|^4 u = 0$

Even in this special setting one is only able to construct few almost-periodic solutions.

few $\rightsquigarrow$ special and/or very high regularity!

Authors	Decay of $\lvert u_j \rvert$	Regularity of V
[Bo'96]	at least superexponential	analytic
[Pö'02]	at least superexponential	ℓ_2
[Geng-Xu'12–'16]	exponential	ℓ_2
[Bo'04 + below*]	subexponential	ℓ_∞
[BMP'22]	polynomial	ℓ_∞
[CGP'23]	subexponential	h_p

below* = Cong-Liu-Shi-Yuan, Biasco-Massetti-P., Cong-Mi-Shi-Wu, Cong-Yuan (NLW), Cong, Cong-Wu high dimension

Finite regularity solutions for NLS

$$\mathrm{i}u_t + u_{xx} - V * u + F(|u|^2)u = 0 \qquad (1)$$

Theorem (Biasco-Massetti-P. 22)

For any $p > 1$ and for most choices of $V \in \ell^\infty$ there exist infinitely many (both weak and classical) almost periodic solutions

$$u(t,x) = \sum_j \hat{u}_j(t)e^{\mathrm{i}jx}\,, \quad \omega_j \sim j^2\,,\ \sup_j |\hat{u}_j||j|^p \ll 1\,.$$

Such solutions are approximately supported on sparse subsets of $\mathbb{Z}$.

For example Set $\mathcal{S} := \{j \in \mathbb{Z} : j = 2^h\,, \quad h \in \mathbb{N}\}$

we have a solution with $|\hat{u}_j| \sim |j|^{-p}$ for all $j \in \mathcal{S}$

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Open problems

- Find Maximal tori (no conditions on $\mathcal{S}$) of finite regularity.
- NLS with multiplicative potential

$$\mathrm{i}u_t - u_{xx} + V(x)u + |u|^4 u = 0$$

- Degenerate KAM theory

$$u_{tt} + u_{xxxx} + mu + u^3 = 0$$

- Fix some (possibly very strong) regularity class, and prove that for most potentials $V \in \ell_\infty$ typical solutions in that class are almost-periodic.

Thanks!