From moments to sparse representations, a geometric, algebraic and algorithmic viewpoint

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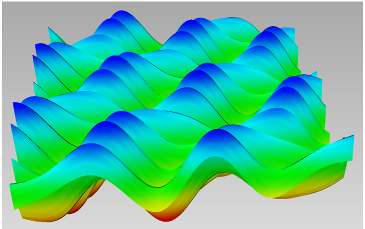
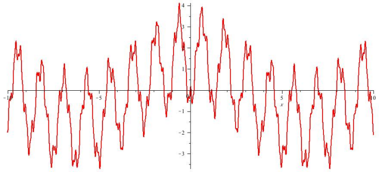
Part I



Sparse representation of signals

Given a function or signal $f(t)$:



decompose it as

$$f(t) = \sum_{i=1}^{r'} (a_i cos(\mu_i t) + b_i sin(\mu_i t)) e^{\nu_i t} = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$$

Prony's method (1795)

For the signal $f(t) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$, $(\omega_i, \zeta_i \in \mathbb{C})$,

- Evaluate f at $2\,r$ regularly spaced points: $\sigma_0 := f(0), \sigma_1 := f(1), \ldots$
- Compute a non-zero element $\mathbf{p} = [\mathbf{p_0}, \ldots, \mathbf{p_r}]$ in the kernel:

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_{\mathbf{1}} = \mathbf{e}^{\zeta_1}, \ldots, \xi_{\mathbf{r}} = \mathbf{e}^{\zeta_r}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.
- Solve the system

$$\begin{bmatrix} 1 & \ldots & \ldots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \ldots & \ldots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_{\mathbf{1}} \\ \omega_{\mathbf{2}} \\ \vdots \\ \omega_{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}.$$

Symmetric tensor decomposition and Waring problem (1770)

Symmetric tensor decomposition problem:

Given a homogeneous polynomial ψ of degree d in the variables $\overline{\mathbf{x}} = (x_0, x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

$$\psi(\overline{\mathbf{x}}) = \sum_{|\alpha|=d} \sigma_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha,$$

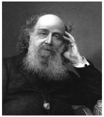
find a minimal decomposition of ψ of the form

$$\psi(\overline{\mathbf{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0}x_0 + \xi_{i,1}x_1 + \cdots + \xi_{i,n}x_n)^d$$

with $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^{n+1}$ spanning disctint lines, $\omega_i \in \overline{\mathbb{K}}$.

The minimal r in such a decomposition is called the **rank** of ψ.

Sylvester approach (1851)



Theorem

The binary form $\psi(x_0, x_1) = \sum_{i=0}^{d} \sigma_i \binom{d}{i} x_0^{d-i} x_1^i$ *can be decomposed as a sum of* r *distinct powers of linear forms*

$$\psi = \sum_{k=1}^{r} \omega_k (\alpha_{\mathbf{k}} x_0 + \beta_{\mathbf{k}} x_1)^d$$

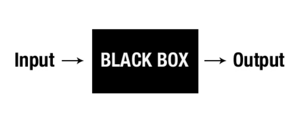
iff there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{d-r} & \dots & \sigma_{d-1} & \sigma_d \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

and of the form $p = c \prod_{k=1}^{r} (\beta_{\mathbf{k}} x_0 - \alpha_{\mathbf{k}} x_1)$ *with* $(\alpha_k : \beta_k)$ *distinct.*

Sparse interpolation

Given a black-box polynomial function $f(x)$



find what are the terms inside from output values.

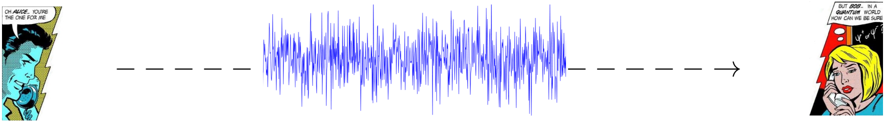
☞ Find $r \in \mathbb{N}, \omega_i \in \mathbb{C}, \alpha_i \in \mathbb{N}$ such that $f(x) = \sum_{i=1}^{r} \omega_i \, x^{\alpha_i}$.

- Choose $\varphi \in \mathbb{C}$
- Compute the sequence of terms $\sigma_0 = f(1), \ldots, \sigma_{2r-1} = f(\varphi^{2r-1})$;
- Construct the matrix $H = [\sigma_{i+j}]$ and its kernel $p = [p_0, \ldots, p_r]$ s.t.

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_{\mathbf{1}} = \varphi^{\alpha_1}, \ldots, \xi_{\mathbf{r}} = \varphi^{\alpha_{\mathbf{r}}}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$ and deduce the exponents $\alpha_{\mathbf{i}} = \log_\varphi(\xi_{\mathbf{i}})$.
- Deduce the weights $W = [\omega_i]$ by solving $V_\Xi W = [\sigma_0, \ldots, \sigma_{r-1}]$ where V_Ξ is the Vandermonde system of the roots $\xi_1, \ldots, \xi_r$.

Decoding



An algebraic code:

$$E = \{c(f) = [f(\xi_1), \dots, f(\xi_m)] \mid f \in \mathbb{K}[x]; \deg(f) \le d\}.$$

Encoding messages using the dual code:

$$C = E^{\perp} = \{\mathbf{c} \mid \mathbf{c} \cdot [f(\xi_1), \dots, f(\xi_l)] = 0 \ \forall f \in V = \langle \mathbf{x}^{\mathbf{a}} \rangle \subset \mathbb{F}[\mathbf{x}]\}$$

Message received: $r = m + e$ for $m \in C$ where $e = [\omega_1, \dots, \omega_m]$ is an error with $\omega_j \neq 0$ for $j = i_1, \dots, i_r$ and $\omega_j = 0$ otherwise.

☞ **Find the error e.**

Berlekamp-Massey method (1969)

- Compute the syndrome $\sigma_k = c(x^k) \cdot r = c(x^k) \cdot e = \sum_{j=1}^{r} \omega_{i_j} \xi_{i_j}^k$.
- Compute the matrix

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \dots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

  and its kernel $p = [p_0, \dots, p_r]$.
- Compute the roots of the error locator polynomial $p(x) = \sum_{i=0}^{r} p_i \, x^i = p_r \prod_{j=1}^{r} (x - \xi_{i_j})$.
- Deduce the errors ω_{i_j}.

Simultaneous decomposition

Simultaneous decomposition problem

Given symmetric tensors $\psi_1, \dots, \psi_m$ of order $d_1, \dots, d_m$, find a simultaneous decomposition of the form

$$\psi_l = \sum_{i=1}^{r} \omega_{l,i} (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^{d_l}$$

where $\xi_i = (\xi_{i,0}, \dots, \xi_{i,n})$ span distinct lines in $\overline{\mathbb{K}}^{n+1}$ and $\omega_{l,i} \in \overline{\mathbb{K}}$ for $l = 1, \dots, m$.

Proposition (One dimensional decomposition)

Let $\psi_l = \sum_{i=0}^{d_l} \sigma_{1,i} \binom{d_l}{i} x_0^{d_l - i} x_1^i \in \mathbb{K}[x_0, x_1]_{d_l}$ *for* $l = 1, \ldots, m$.
If there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\left[\begin{array}{cccc} \sigma_{1,0} & \sigma_{1,1} & \cdots & \sigma_{1,r} \\ \sigma_{1,1} & & & \sigma_{1,r+1} \\ \vdots & & & \vdots \\ \sigma_{1,d_1-r} & \cdots & \sigma_{1,d_1-1} & \sigma_{1,d_1} \\ \hline \vdots & & & \vdots \\ \hline \sigma_{m,0} & \sigma_{m,1} & \cdots & \sigma_{m,r} \\ \sigma_{m,1} & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{m,d_m-r} & \cdots & \sigma_{m,d_m-1} & \sigma_{m,d_m} \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ \vdots \\ p_r \end{array}\right] = 0$$

of the form $p = c \prod_{k=1}^{r} (\beta_k x_0 - \alpha_k x_1)$ *with* $[\alpha_k : \beta_k]$ *distinct, then*

$$\psi_l = \sum_{i=1}^{r} \omega_{i,l} (\alpha_l x_0 + \beta_l x_1)^{d_l}$$

for $\omega_{i,l} \in \overline{\mathbb{K}}$ *and* $l = 1, \ldots, m$.

Sequences, series, duality (1D)

Sequences: $\sigma = (\sigma_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ indexed by $k \in \mathbb{N}$.

Formal power series:

$$\sigma(y) = \sum_{k=0}^{\infty} \sigma_k \frac{y^k}{k!} \in \mathbb{K}[[y]] \qquad \sigma(\mathbf{z}) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \sigma_{\mathbf{k}} \, \mathbf{z}^{\mathbf{k}} \in \mathbb{K}[[\mathbf{z}]]$$

Linear functionals: $\mathbb{K}[x]^* = \{\Lambda : \mathbb{K}[x] \to \mathbb{K} \text{ linear}\}$.

Example:

- $p \mapsto$ coefficient of x^i in $p = \frac{1}{i!}\partial^i(p)(0)$
- $\mathfrak{e}_\zeta : p \mapsto p(\zeta)$.

Series as linear functionals: For $\sigma(y) = \sum_{k=0}^{\infty} \sigma_k \frac{y^k}{k!} \in \mathbb{K}[[y]]$ or $\sigma(z) = \sum_{k=0}^{\infty} \sigma_k z^k \in \mathbb{K}[[z]]$,

$$\sigma : p = \sum_k p_k x^k \mapsto \langle \sigma | p \rangle = \sum_k \sigma_k p_k$$

$(\frac{y^k}{k!})$ **(resp.** (z^k)**)** is the dual basis of the monomial basis $(x^k)_{k \in \mathbb{N}}$.

Example:

$$\mathfrak{e}_\zeta(y) = \sum_{k=0}^{\infty} \zeta^k \frac{y^k}{k!} = e^{\zeta y} \in \mathbb{K}[[y]] \quad \mathfrak{e}_\zeta(\mathbf{z}) = \sum_{k=0}^{\infty} \zeta^{\mathbf{k}} \mathbf{z}^{\mathbf{k}} = \frac{1}{1-\zeta \mathbf{z}} \in \mathbb{K}[[\mathbf{z}]]$$

Structure of $\mathbb{K}[x]$-module: $p \star \Lambda : q \mapsto \Lambda(p\, q)$.

$$\begin{aligned} x \star \sigma(y) &= \sum_{k=1}^{\infty} \sigma_k \frac{y^{k-1}}{(k-1)!} = \partial(\sigma(y)) \\ p(x) \star \sigma(y) &= p(\partial)(\sigma(y)) \end{aligned}$$

$$\mathbf{p}(\mathbf{x}) \star \sigma(\mathbf{z}) = \pi_+(\mathbf{p}(\mathbf{z}^{-1})(\sigma(\mathbf{z})))$$

Sequences, series, duality (nD)

Multi-index sequences: $\sigma = (\sigma_\alpha)_{\alpha\in\mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Taylor series:

$$\sigma(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{K}[[y_1, \ldots, y_n]] \qquad \sigma(\mathbf{z}) = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \mathbf{z}^\alpha \in \mathbb{K}[[\mathbf{z}_1, \ldots, \mathbf{z}_n]]$$

where $\alpha! = \prod \alpha_i!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$.

Linear functionals: $\sigma \in R^* = \{\sigma : R \to \mathbb{K}, \textbf{ linear}\}$

$$\sigma : p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mapsto \langle \sigma | p \rangle = \sum_\alpha \sigma_\alpha p_\alpha$$

The coefficients $\langle \sigma | \mathbf{x}^\alpha \rangle = \sigma_\alpha \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are called the **moments** of σ.

Structure of R-module: $\forall p \in R, \sigma \in R^*$, $p \star \sigma : q \mapsto \langle \sigma | p\, q \rangle$:

$$p \star \sigma = p(\partial_1, \ldots, \partial_n)(\sigma)(\mathbf{y}) \qquad \mathbf{p} \star \sigma = \pi_+(\mathbf{p}(\mathbf{z}_1^{-1}, \ldots, \mathbf{z}_n^{-1})\sigma(\mathbf{z}))$$

Symmetric tensor and apolarity

Apolar product: For $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha \in \mathbb{K}[\overline{\mathbf{x}}]_d$,

$$\langle f, g \rangle_d = \sum_{|\alpha|=d} f_\alpha \, g_\alpha \binom{d}{\alpha}.$$

Property: $\langle f, (\xi_0 x_0 + \cdots + \xi_n x_n)^d \rangle = f(\xi_0, \ldots, \xi_n)$

Duality: For $\psi \in S_d$, we define $\psi^* \in S_d^* = \mathrm{Hom}_{\mathbb{K}}(S_d, \mathbb{K})$ as

$$\begin{aligned} \psi^* : S_d &\to \mathbb{K} \\ p &\mapsto \langle \psi, p \rangle_d \end{aligned}$$

Example: $((\xi_0 x_0 + \cdots + \xi_n x_n)^d)^* = \mathfrak{e}_\xi : p \in S_d \mapsto p(\xi)$ (evaluation at ξ)

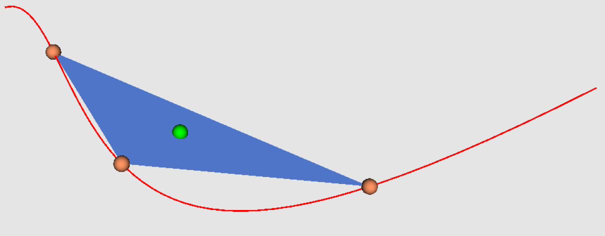
Dual symmetric tensor decomposition problem:

Given $\psi^* \in S_d^*$, find a decomposition of the form $\psi^* = \sum_{i=1}^r \omega_i \mathfrak{e}_{\xi_i}$ where $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n})$ span distinct lines in $\overline{\mathbb{K}}^{n+1}$, $\omega_i \in \overline{\mathbb{K}}$ ($\omega_i \neq 0$).

Symmetric tensors and secants

The evaluation $\mathfrak{e}_\xi \in S_d^*$ at $\xi \in \overline{\mathbb{K}}^{n+1}$ represented by the vector $(\xi^\alpha)_{|\alpha|=d}$ defines a point of the **Veronese** variety $\mathcal{V}_d^n \subset \mathbb{P}(S_d^*)$.

$\psi^* = \sum_{i=1}^r \omega_i \mathfrak{e}_{\xi_i}$ iff the corresponding point $[\psi^*]$ in $\mathbb{P}(S_d^*)$ is in the linear span of the evaluations $[\mathfrak{e}_{\xi_i}] \in \mathcal{V}_d^n$.



Let $S_r^o(\mathcal{V}_d^n) = \{[\psi^*] \in \mathbb{P}(S_d^*) \mid \psi^* = \sum_{i=1}^r \omega_i e_i \text{ with } [e_i] \in \mathcal{V}_d^n, \omega_i \in \mathbb{K}\}$.

The closure $S_r(\mathcal{V}_d^n) = \overline{S_r^o(\mathcal{V}_d^n)}$ is the **r^{th}-secant** of $\mathcal{V}_d^n$.

Dehomogeneization (aparté)

$$S = \mathbb{K}[x_0, \ldots, x_n] \qquad R = \mathbb{K}[x_1, \ldots, x_n]$$

$$\begin{array}{rcl} \iota_0 : p(x_0, \ldots, x_n) & \mapsto & p(1, x_1, \ldots, x_n) \\ x_0^{\deg(p)} p(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) & \leftarrow & p(x_1, \ldots, x_n) : h_0 \end{array}$$

Dual action

- $h_0^* : \sigma \in S_d^* \mapsto \sigma \circ h_0 \in R_{\leq d}^*$
- $\iota_0^* : \sigma \in R_{\leq d}^* \mapsto \sigma \circ \iota \in S_d^*$

$$\iota_0^*(\sigma(\mathbf{y}) + \mathcal{O}(\mathbf{y})^d) = [\mathfrak{e}_0\, \sigma(\mathbf{y})]_d$$

where $\mathfrak{e}_0 = \sum_i \frac{y_0^k}{k!}$.

For $I \subset R$, let $[\mathfrak{e}_0\, I^{\perp}]_*$ be the vector space of homogeneous components of $\mathfrak{e}_0\, \sigma(\mathbf{y})$ for $\sigma \in I^{\perp} \subset R^*$, then

$$[\mathfrak{e}_0\, I^{\perp}]_* = (J : x_0^*)^{\perp}$$

for any J such that $\iota_0(J) = I$ (e.g. $J = (I^{h_0})$).

Inverse systems

For I an ideal in $R = \mathbb{K}[\mathbf{x}]$,

$$I^{\perp} = \{\sigma \in R^* \mid \forall p \in I,\ \langle \sigma | p \rangle = 0\}.$$

- In $\mathbb{K}[[\mathbf{y}]]$, $I^{\perp}$ is stable by **derivations** with respect to y_i.
- **In $\mathbb{K}[[\mathbf{z}]]$, $I^{\perp}$ is stable by "division" by variables z_i.**

Inverse system generated by $\omega_1, \ldots, \omega_r \in \mathbb{K}[\mathbf{y}]$

$$\langle\langle \omega_1, \ldots, \omega_r \rangle\rangle = \langle \partial_{\mathbf{y}}^{\alpha}(\omega_i), \alpha \in \mathbb{N}^n \rangle \quad \text{resp. } \langle \pi_+(\mathbf{z}^{-\alpha}\omega_{\mathbf{i}}(\mathbf{z})), \alpha \in \mathbb{N}^{\mathbf{n}} \rangle$$

Example: $I = (x_1^2, x_2^2) \subset \mathbb{K}[x_1, x_2]$

$$I^{\perp} = \langle 1, y_1, y_2, y_1 y_2 \rangle = \langle\langle y_1 y_2 \rangle\rangle \quad \text{resp. } \langle \mathbf{1}, \mathbf{z_1}, \mathbf{z_2}, \mathbf{z_1 z_2} \rangle = \langle\langle \mathbf{z_1 z_2} \rangle\rangle$$

Dual of quotient algebra: for $\mathcal{A} = R/I$, $\mathcal{A}^* = I^{\perp}$.

Hankel operators

Hankel operator: For $\sigma = (\sigma_1, \ldots, \sigma_m) \in (R^*)^m$,

$$\begin{array}{rcl} H_\sigma : R & \to & (R^*)^m \\ p & \mapsto & (p \star \sigma_1, \ldots, p \star \sigma_m) \end{array}$$

σ is the **symbol** of H_σ.

Truncated Hankel operator: $V, W_1, \ldots, W_m \subset R$,

$$H_\sigma^{W,V} : p \in V \to ((p \star \sigma_i)_{|W_i})$$

Example: $V = \langle \mathbf{x}^\alpha, \alpha \in A\rangle = \langle \mathbf{x}^A\rangle, W = \langle \mathbf{x}^\beta, \beta \in B\rangle = \langle \mathbf{x}^B\rangle \subset R$, $\sigma \in R^*$,

$$H_\sigma^{A,B} = [\langle \sigma | \mathbf{x}^\alpha \mathbf{x}^\beta\rangle]_{\alpha \in A, \beta \in B} = [\sigma_{\alpha+\beta}]_{\alpha \in A, \beta \in B}.$$

Ideal:

$$\begin{aligned} I_\sigma = \ker H_\sigma &= \{p \in \mathbb{K}[\mathbf{x}] \mid p \star \sigma = 0\}, \\ &= \{p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mid \forall \beta \in \mathbb{N}^n \sum_\alpha p_\alpha \sigma_{\alpha+\beta} = 0\} \end{aligned}$$

Linear recurrence relations on the sequence $\sigma = (\sigma_\alpha)_{\alpha \in \mathbb{N}^n}$.

Quotient algebra: $\mathcal{A}_\sigma = R/I_\sigma$ ☞ **Studied case:** $\dim \mathcal{A}_\sigma < \infty$

Structure of an Artinian algebra $\mathcal{A}$

Definition: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ is **Artinian** if $\dim_{\mathbb{K}} \mathcal{A} < \infty$.

Hilbert nullstellensatz: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian $\Leftrightarrow \mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$ is finite.

Assuming $\mathbb{K} = \overline{\mathbb{K}}$ is algebraically closed, we have

- $I = Q_1 \cap \cdots \cap Q_r$ where Q_i is m_{ξ_i}-primary where $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$.
- $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$, with
  - $\mathcal{A}_i = \mathbf{u}_i\, \mathcal{A} \sim \mathbb{K}[x_1, \ldots, x_n]/Q_i$,
  - $\mathbf{u}_i^2 = \mathbf{u}_i$, $\mathbf{u}_i\, \mathbf{u}_j = 0$ if $i \neq j$, $\mathbf{u}_1 + \cdots + \mathbf{u}_r = 1$.
- $\dim R/Q_i = \mu_i$ is the multiplicity of ξ_i.

Structure of the dual $\mathcal{A}^*$

Sparse series:

$$\mathcal{P}ol\mathcal{E}xp = \left\{ \sigma(\mathbf{y}) = \sum_{i=1}^{r} \omega_i(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \mid \omega_i(\mathbf{y}) \in \mathbb{K}[\mathbf{y}], \right\}$$

where $\mathfrak{e}_{\xi_i}(\mathbf{y}) = e^{\mathbf{y}\cdot\xi_i} = e^{y_1\xi_{1,i}+\cdots+y_n\xi_{n,i}}$ with $\xi_{i,j} \in \mathbb{K}$.

Inverse system generated by $\omega_1, \ldots, \omega_r \in \mathbb{K}[\mathbf{y}]$

$$\langle\langle \omega_1, \ldots, \omega_r \rangle\rangle = \langle \partial_{\mathbf{y}}^{\alpha}(\omega_i), \alpha \in \mathbb{N}^n \rangle$$

Theorem

For $\mathbb{K} = \overline{\mathbb{K}}$ algebraically closed,

$$\mathcal{A}^* = \oplus_{i=1}^{r} \mathcal{D}_i\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \quad \subset \quad \mathcal{P}ol\mathcal{E}xp$$

- $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$
- $\mathcal{D}_i = \langle\langle \omega_{i,1}, \ldots, \omega_{i,l_i} \rangle\rangle$ *with* $\omega_{i,j} \in \mathbb{K}[\mathbf{y}]$, $Q_i^{\perp} = \mathcal{D}_i \mathfrak{e}_{\xi}$ *where* $I = Q_1 \cap \cdots \cap Q_r$
- $\mu(\omega_{i,1}, \ldots, \omega_{i,l_i}) := \dim_{\mathbb{K}}(\mathcal{D}_i) = \mu_i$ *multiplicity of* ξ_i.

The roots by eigencomputation

Hypothesis: $\mathcal{V}_{\mathbb{K}}(I) = \{\xi_1, \ldots, \xi_r\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian.

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathsf{t}} : \mathcal{A}^* & \to & \mathcal{A}^* \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of $\mathcal{M}_a$ are $\{a(\xi_1), \ldots, a(\xi_r)\}$.*
- *The eigenvectors of all $(\mathcal{M}_a^{\mathsf{t}})_{a \in \mathcal{A}}$ are (up to a scalar) $\mathbf{e}_{\xi_i} : p \mapsto p(\xi_i)$.*

Proposition

If the roots are simple, the operators $\mathcal{M}_a$ are diagonalizable. Their common eigenvectors are, up to a scalar, interpolation polynomials $\mathbf{u}_i$ at the roots and idempotent in $\mathcal{A}$.

Example

Roots of polynomial systems

$$\left\{ \begin{array}{lcl} f_1 & = & x_1^2 x_2 - x_1^2 \\ f_2 & = & x_1 x_2 - x_2 \end{array} \right. \qquad I = (f_1, f_2) \subset \mathbb{C}[\mathbf{x}]$$

$$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I \equiv \langle 1, x_1, x_2 \rangle \quad I = (x_1^2 - x_2, x_1 x_2 - x_2, x_2^2 - x_2)$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{array}{c} \text{common} \\ \text{eigvecs of} \\ M_1^t, M_2^t \end{array} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$I = Q_1 \cap Q_2 \quad \text{where} \quad Q_1 = (x_1^2, x_2), \quad Q_2 = \mathbf{m}_{(1,1)} = (x_1 - 1, x_2 - 1)$$

$$I = Q_1^\perp \oplus Q_2^\perp \quad Q_1^\perp = \langle 1, y_1 \rangle = \langle 1, y_1 \rangle\, \mathfrak{e}_{(0,0)}(\mathbf{y}) \quad Q_2^\perp = \langle 1 \rangle\, \mathfrak{e}_{(1,1)}(\mathbf{y}) = \langle e^{y_1 + y_2} \rangle$$

Solution of partial differential equations (with constant coeff.)

$$\left\{ \begin{array}{lcl} \partial_{y_1}^2 \partial_{y_2} \sigma - \partial_{y_1}^2 \sigma & = & 0 \\ \partial_{y_1} \partial_{y_2} \sigma - \partial_{y_2} \sigma & = & 0 \end{array} \right. \qquad \begin{array}{l} f_1 \star \sigma = 0 \\ f_2 \star \sigma = 0 \end{array} \quad \Rightarrow \quad \sigma \in I^\perp = Q_1^\perp \oplus Q_2^\perp$$

$$\sigma = a + b\, y_1 + c\, e^{y_1 + y_2} \quad a, b, c \in \mathbb{C}$$

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