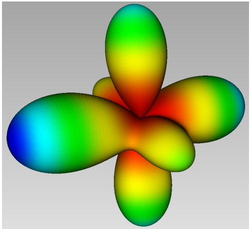
From moments to sparse representations, a geometric, algebraic and algorithmic viewpoint

Bernard Mourrain
Inria Méditerranée, Sophia Antipolis
Bernard.Mourrain@inria.fr

Part II



Univariate series:

Kronecker (1881)

The Hankel operator

$$\begin{array}{rcl} H_\sigma : \mathbb{C}^{\mathbb{N},finite} & \rightarrow & \mathbb{C}^{\mathbb{N}} \\ (p_m) & \mapsto & (\sum_m \sigma_{m+n} p_m)_{n\in\mathbb{N}} \end{array}$$

is of **finite rank** r iff $\exists \omega_1, \ldots, \omega_{r'} \in \mathbb{C}[y]$ and $\xi_1, \ldots, \xi_{r'} \in \mathbb{C}$ distincts s.t.

$$\sigma(y) = \sum_{n\in\mathbb{N}} \sigma_n \frac{y^n}{n!} = \sum_{i=1}^{r'} \omega_i(y) \mathbf{e}_{\xi_i}(y)$$

with $\sum_{i=1}^{r'} (\deg(\omega_i) + 1) = r$.

Multivariate series:

Theorem (Generalized Kronecker Theorem)

For $\sigma = (\sigma_1, \ldots, \sigma_m) \in (R^*)^m$, the Hankel operator

$$\begin{array}{rcl} H_\sigma : R & \to & (R^*)^m \\ p & \mapsto & (p \star \sigma_1, \ldots, p \star \sigma_m) \end{array}$$

is of rank r iff

$$\sigma_j = \sum_{i=1}^{r'} \omega_{j,i}(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \in \mathcal{P}ol\mathcal{E}xp, \quad j = 1, \ldots, m$$

with $r = \sum_{i=1}^{r'} \mu(\omega_{1,i}, \ldots, \omega_{m,i})$. In this case, we have

- $\mathcal{V}_{\mathbb{C}}(I_\sigma) = \{\xi_1, \ldots, \xi_{r'}\}$.
- $I_\sigma = Q_1 \cap \cdots \cap Q_{r'}$ *with* $Q_i^\perp = \langle\langle \omega_{1,i}, \ldots, \omega_{m,i} \rangle\rangle\, \mathfrak{e}_{\xi_i}(\mathbf{y})$.

If $m = 1$, $\mathcal{A}_\sigma$ is **Gorenstein** *($\mathcal{A}_\sigma^* = \mathcal{A}_\sigma \star \sigma$ is a free $\mathcal{A}_\sigma$-module of rank 1) and $(a, b) \mapsto \langle \sigma | ab \rangle$ is non-degenerate in $\mathcal{A}_\sigma$.*

The decomposition from the algebraic structure

Decomposition problem

Given a (truncated) sequence of moments σ_α, $\alpha \in A$, find
$\xi_i = (\xi_{i,1}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^n$ disctint, $\omega_i \in \overline{\mathbb{K}}$. s.t. $\sigma = \sum_i \omega_i \mathfrak{e}_{\xi_i}$

Hankel operator: For $\sigma \in R^*$,

$$\begin{array}{rcl} H_\sigma : R & \rightarrow & R^* \\ p & \mapsto & p \star \sigma \end{array}$$

Quotient algebra: $\mathcal{A}_\sigma = R/I_\sigma$ where $I_\sigma = \ker H_\sigma$.

$$\begin{array}{rcl} 0 \rightarrow I_\sigma \rightarrow \mathbb{K}[\mathbf{x}] & \xrightarrow{H_\sigma} & \mathcal{A}_\sigma^* \rightarrow 0 \\ p & \mapsto & p \star \sigma \end{array}$$

Isomorphism between $\mathcal{A}_\sigma$ and $\mathcal{A}_\sigma^* = I_\sigma^\perp$.
($\mathcal{A}_\sigma$ **Gorenstein**, i.e. $\exists \tau = \sigma \in \mathcal{A}_\sigma^*$ s.t. $\mathcal{A}_\sigma^* = \mathcal{A}_\sigma \star \tau$ is a free $\mathcal{A}_\sigma$-module).
☞ **Find the points ξ_i as the roots of I_σ and the weights ω_i from the idempotents of $\mathcal{A}_\sigma$.**

The structure of $\mathcal{A}_\sigma$

For $\sigma = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}}\, \mathfrak{e}_{\xi_{\mathbf{i}}}$, with $\omega_{\mathbf{i}} \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct.

- rank $H_\sigma = \mathbf{r}$ and the multiplicity of the points $\xi_{\mathbf{1}}, \ldots, \xi_{\mathbf{r}}$ in $\mathcal{V}(I_\sigma)$ is 1.
- For B, B' be of size r, $H_\sigma^{B',B}$ invertible iff B and B' are bases of $\mathcal{A}_\sigma = \mathbb{K}[\mathbf{x}]/I_\sigma$.
- The matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}_\sigma$ is such that

$$\mathbf{H}_\sigma^{\mathbf{B}',\mathbf{x_i}\mathbf{B}} = \mathbf{H}_{\mathbf{x_i}\star\sigma}^{\mathbf{B}',\mathbf{B}} = \mathbf{H}_\sigma^{\mathbf{B}',\mathbf{B}}\, \mathbf{M_i}$$

- The common **eigenvectors** of M_i are (up to a scalar) the Lagrange **interpolation polynomials** $\mathbf{u}_{\xi_{\mathbf{i}}}$ at the points $\xi_{\mathbf{i}}$, $i = 1, \ldots, r$.

$$\mathbf{u}_{\xi_{\mathbf{i}}}(\xi_{\mathbf{j}}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad \mathbf{u}_{\xi_{\mathbf{i}}}{}^2 \equiv \mathbf{u}_{\xi_{\mathbf{i}}},\ \sum_{i=1}^{\mathbf{r}} \mathbf{u}_{\xi_{\mathbf{i}}} \equiv 1.$$

- The common **eigenvectors** of M_i^t are (up to a scalar) the vectors $[B(\xi_{\mathbf{i}})]$, $i = 1, \ldots, r$.

Decomposition algorithm

Input: The first coefficients $(\sigma_\alpha)_{\alpha \in A}$ of the series

$$\sigma = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}(\mathbf{y})$$

1. Compute bases $B, B' \subset \langle \mathbf{x}^A \rangle$ s.t. that $H^{B',B}$ invertible and $|B| = |B'| = r = \dim \mathcal{A}_\sigma$;
2. Deduce the tables of multiplications $M_i := (H_\sigma^{B',B})^{-1} H_\sigma^{B',x_i B}$
3. Compute the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_r$ of $\sum_i l_i M_i$ for a generic $\mathbf{l} = l_1 x_1 + \cdots + l_n x_n$;
4. Deduce the points $\xi_i = (\xi_{i,1}, \dots, \xi_{i,n})$ s.t. $M_j \mathbf{v}_i - \xi_{i,j} \mathbf{v}_i = 0$ and the weights $\omega_i = \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle$.

Output: The decomposition $\sigma = \sum_{i=1}^{r} \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle \mathbf{e}_{\xi_i}(\mathbf{y})$.

Demo

Multivariate Prony method (1)

Let $h(t_1, t_2) = 2 + 3\, 2^{t_1}\, 2^{t_2} - 3^{t_1}$, $\sigma = \sum_{\alpha \in \mathbb{N}^2} h(\alpha) \frac{\mathbf{y}^\alpha}{\alpha!} = 2\mathfrak{e}_{(1,1)}(\mathbf{y}) + 3\mathfrak{e}_{(2,2)}(\mathbf{y}) - \mathfrak{e}_{(3,1)}(\mathbf{y})$.

- Take $B = \{1, x_1, x_2\}$ and compute

$$H_0 := H_\sigma^{B,B} = \begin{bmatrix} h(0,0) & h(1,0) & h(0,1) \\ h(1,0) & h(2,0) & h(1,1) \\ h(0,1) & h(1,1) & h(0,2) \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 \\ 5 & 5 & 11 \\ 7 & 11 & 13 \end{bmatrix},$$

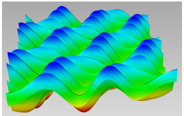
$$H_1 := H_\sigma^{B,x_1\,B} = \begin{bmatrix} 5 & 5 & 7 \\ 5 & -1 & 17 \\ 811 & 178 & 23 \end{bmatrix}, H_2 := H_\sigma^{B,x_2\,B} = \begin{bmatrix} 7 & 11 & 13 \\ 11 & 17 & 23 \\ 13 & 23 & 25 \end{bmatrix}.$$

- Compute the generalized eigenvectors of $(aH_1 + bH_2, H_0)$:

$$U = \begin{bmatrix} 2 & -1 & 0 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix} \text{ and } H_0\, U = \begin{bmatrix} 2 & 3 & -1 \\ 2 \times 1 & 3 \times 2 & -1 \times 3 \\ 2 \times 1 & 3 \times 2 & -1 \times 1 \end{bmatrix}.$$

- This yields the weights $2, 3, -1$ and the roots $(1,1), (2,2), (3,1)$.

Multivariate Prony method (2)



$h(t_1, t_2) := \sum_{i=1}^{r} \omega_i e^{f_1 t_1 + f_2 t_2}$ with

$$\mathbf{f} := \begin{bmatrix} -0.5 & 1.0 + 3.141592654\,i \\ 0.1 + 21.36283005\,i & 1.5 + 32.67256360\,i \\ 0.1 + 21.36283005\,i & -0.5 + 79.16813488\,i \\ -2.5 + 145.7698991\,i & -10.0 + 517.1061508\,i \end{bmatrix} \omega := \begin{bmatrix} 1.375328890 + 0.9992349291\,i \\ 1.046162168 + 0.3399186938\,i \\ 0.9 \\ -9.2 \end{bmatrix}$$

For the sampling $[\frac{1}{50}, \frac{1}{170}]$, $B = \{1, x_1, x_2, x_1 x_2\}$, the SVD of $H_\sigma^{B,B}$ is

$$[33.1196344300301391, 14.3767453860219057, 0.244096952193142480, 0.0230734326225932214]$$

and the computed decomposition is

$$\mathbf{f}^* = \begin{bmatrix} -2.49999999703636711 + 145.769899153890435\,i & -9.9999999913514852 + 517.106150711515852\,i \\ 0.0999940670173818935 + 21.3628392917863437\,i & -0.500045063743692286 + 79.1681566575291527\,i \\ 0.100028305341504586 + 21.3628527756206275\,i & 1.50002358381760881 + 32.6725933609709571\,i \\ -0.499926454593063452 + 0.0000142466247443506387\,i & 1.00008814016387371 + 3.14161379568963772\,i \end{bmatrix}$$

$$\omega^* = \begin{bmatrix} -9.19999999613861696 - 0.00000000772422142913953280\,i \\ 0.899999936743709261 - 0.0000015620281484940434 8\,i \\ 1.04615643213670850 + 0.339923495269889020\,i \\ 1.37533468654902213 + 0.999231697828891208\,i \end{bmatrix}$$

Sparse interpolation

$$f(\mathbf{x}) = \sum_{i=1}^{r} \omega_i\, \mathbf{x}^{\alpha_i} \quad \Rightarrow \quad \sigma = \sum_{\gamma} f(\varphi^{\gamma})\, \frac{\mathbf{y}^{\gamma}}{\gamma!} = \sum_{i=1}^{r} \omega_i\, \mathbf{e}_{\varphi^{\alpha_i}}(\mathbf{y})$$

Example: $f(x_1, x_2) = x_1^{33} x_2^{12} - 5\, x_1 x_2^{45} + 101.$

- Compute $\sigma_\alpha = f(\varphi_1^{\alpha_1}, \varphi_2^{\alpha_2})$ for $\alpha_1 + \alpha_2 \leq 3$ and $\varphi_1 = \varphi_2 = e^{\frac{2\mathbf{i}\pi}{50}}$.
- Compute the Hankel matrix $H_\sigma^{1,2}$:

$$\begin{bmatrix} 97.00000 & 97.01771 + 3.93695\,\mathbf{i} & 95.50360 - 1.47099\,\mathbf{i} & 98.46280 + 4.88062\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 9 \\ 97.01771 + 3.93695\,\mathbf{i} & 98.46280 + 4.88062\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 102.35770 + 3.77300\,\mathbf{i} & 99.50853 + 5.29465\,\mathbf{i} & 9 \\ 95.50360 - 1.47099\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 95.73130 - .33862\,\mathbf{i} & 99.50853 + 5.29465\,\mathbf{i} & 95.42134 + 1.47250\,\mathbf{i} & 9 \end{bmatrix}$$

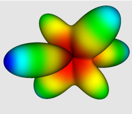
- Deduce the decomposition of $\sigma = \sum_{i=1}^{3} \omega_i \mathbf{e}_{\xi_i}$:

$$\Xi = \begin{bmatrix} 0.99211 + 0.12533\mathbf{i} & 0.80902 - 0.58779\mathbf{i} \\ 1.00000 + 4.86234e^{-11}\mathbf{i} & 1.00000 - 6.91726e^{-10}\mathbf{i} \\ -0.53583 - 0.84433\mathbf{i} & 0.06279 + 0.99803\mathbf{i} \end{bmatrix} \omega = \begin{bmatrix} -5.00000 - 4.43772e^{-7}\mathbf{i} \\ 101.00000 + 4.65640e^{-7}\mathbf{i} \\ 1.00000 - 1.92279e^{-8}\mathbf{i} \end{bmatrix}$$

- and the exponents $\frac{50\,\Xi}{2\pi i}$ mod 50 of the terms of f:

$$\begin{bmatrix} 1.00000 - 0.414119e^{-7}\,\mathbf{i} & -5.00000 + 0.270858e^{-6}\,\mathbf{i}, \\ 0.386933e^{-9} + 0.137963e^{-8}\,\mathbf{i} & -0.550458e^{-8} - 0.38761e^{-8}\,\mathbf{i} \\ -17.00000 - 0.100085e^{-6}\,\mathbf{i} & 12.00000 + 0.700984e^{-6}\,\mathbf{i} \end{bmatrix}$$

Symmetric tensor decomposition



$$\begin{aligned}
\tau &= (\mathbf{x_0} - \mathbf{x_1} + 3\,\mathbf{x_2})^4 + (\mathbf{x_0} + \mathbf{x_1} + \mathbf{x_2})^4 - 3\,(\mathbf{x_0} + 2\,\mathbf{x_1} + 2\,\mathbf{x_2})^4 \\
&= -x_0{}^4 - 24\,x_0{}^3x_1 - 8\,x_0{}^3x_2 - 60\,x_0{}^2x_1{}^2 - 168\,x_0{}^2x_1x_2 - 12\,x_0{}^2x_2{}^2 \\
&\quad -96\,x_0x_1{}^3 - 240\,x_0x_1{}^2x_2 - 384\,x_0x_1x_2{}^2 + 16\,x_0x_2{}^3 - 46\,x_1{}^4 - 200\,x_1{}^3x_2 \\
&\quad -228\,x_1{}^2x_2{}^2 - 296\,x_1x_2{}^3 + 34\,x_2{}^4 \\
\tau^* &= \mathfrak{e}_{(-1,3)}(\mathbf{y}) + \mathfrak{e}_{(1,1)}(\mathbf{y}) - 3\mathfrak{e}_{(2,2)}(\mathbf{y}) \qquad (\textit{by apolarity})
\end{aligned}$$

$$H_{\tau^*}^{2,2} := \begin{bmatrix} -1 & -2 & -6 & -2 & -14 & -10 \\ -2 & -2 & -14 & 4 & -32 & -20 \\ -6 & -14 & -10 & -32 & -20 & -24 \\ -2 & 4 & -32 & 34 & -74 & -38 \\ -14 & -32 & -20 & -74 & -38 & -50 \\ -10 & -20 & -24 & -38 & -50 & -46 \end{bmatrix}$$

For $B = \{1, x_2, x_1\}$,

$$H_{\tau^*}^{B,B} = \begin{bmatrix} -1 & -2 & -6 \\ -2 & -2 & -14 \\ -6 & -14 & -10 \end{bmatrix}, H_{\tau^*}^{B,x_1B} = \begin{bmatrix} -6 & -14 & -10 \\ -14 & -32 & -20 \\ -10 & -20 & -24 \end{bmatrix}, H_{\tau^*}^{B,x_2B} = \begin{bmatrix} -2 & -2 & -14 \\ -2 & 4 & -32 \\ -14 & -32 & -20 \end{bmatrix}$$

- The matrix of multiplication by x_1 in $B = \{1, x_2, x_1\}$ is

$$M_1 = (H_{\tau^*}^{B,B})^{-1} H_{\tau^*}^{B,x_1 B} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

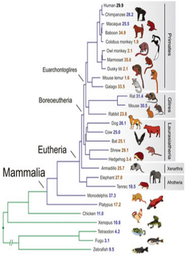
that is the polynomials
$U(x) = \left[\ \frac{1}{2}x_2 - \frac{1}{2}x_1 \quad -2 + \frac{3}{4}x_2 + \frac{1}{4}x_1 \quad -1 + \frac{1}{2}x_2 + \frac{1}{2}x_1\ \right]$.

- We deduce the weights and the frequencies:

$$H_{\tau^*}^{[1,x_1,x_2],U} = \begin{bmatrix} 1 & 1 & -3 \\ 1 \times -1 & 1 \times 1 & -3 \times 2 \\ 1 \times 3 & 1 \times 1 & -3 \times 2 \end{bmatrix}.$$

Weights: $1, 1, -3$; frequencies: $(-1, 3), (1, 1), (2, 2)$.
Decomposition: $\tau^*(\mathbf{y}) = \mathfrak{e}_{(-1,3)}(\mathbf{y}) + \mathfrak{e}_{(1,1)}(\mathbf{y}) - 3\,\mathfrak{e}_{(2,2)}(\mathbf{y}) + (\mathbf{y})^4$

Phylogenetic trees



Problem: study probability vectors for genes $[A, C, G, T]$ and the transitions described by Markov matrices M^i.

Example:

$$\begin{array}{rcccc} \textit{Ancestor} : & & \mathcal{A} & \\ \textit{Transitions} : & M^1 & M^2 & M^3 \\ \textit{Species} : & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 \end{array}$$

For $i_1, i_2, i_3 \in \{A, C, G, T\}$, the probability to observe i_1, i_2, i_3 is

$$p_{i_1,i_2,i_3} = \sum_{k=1}^{4} \pi_k \, M^1_{k,i_1} M^2_{k,i_2} M^3_{k,i_3} \Leftrightarrow \mathbf{p} = \sum_{\mathbf{k=1}}^{\mathbf{4}} \pi_{\mathbf{k}} \, \mathbf{u_k} \otimes \mathbf{v_k} \otimes \mathbf{w_k}$$

where $\mathbf{u}_k = (M^1_{k,1}, \ldots, M^1_{k,4}), \mathbf{v}_k = (M^2_{k,1}, \ldots, M^2_{k,4}), \mathbf{w}_k = (M^3_{k,1}, \ldots, M^3_{k,4})$.

- ☞ p is a tensor $\in \mathbb{K}^4 \otimes \mathbb{K}^4 \otimes \mathbb{K}^4$ of rank ≤ 4.
- ☞ Its decomposition yields the M^i and the ancestor probability (π_j).

A general framework

- $\mathfrak{F}$ the functional space, in which the "signal" lives.
- $S_1, \ldots, S_n : \mathfrak{F} \to \mathfrak{F}$ commuting linear operators: $S_i \circ S_j = S_j \circ S_i$.
- $\Delta : h \in \mathfrak{F} \mapsto \Delta[h] \in \mathbb{C}$ a linear functional on $\mathfrak{F}$.

Generating series associated to $h \in \mathfrak{F}$:

$$\sigma_h(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \Delta[S^\alpha(h)] \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!}.$$

- Eigenfunctions:

$$S_j(E) = \xi_j E, j = 1, \ldots, n \Rightarrow \sigma_E = \omega\, \mathfrak{e}_\xi(\mathbf{y}).$$

- Generalized eigenfunctions:

$$S_j(E_k) = \xi_j E_k + \sum_{k' < k} m_{j,k'} E_{k'} \Rightarrow \sigma_{E_k} = \omega_i(\mathbf{y}) \mathfrak{e}_\xi(\mathbf{y}).$$

☞ **If $h \mapsto \sigma_h$ is injective $\Rightarrow$ unique decomposition of f as a linear combination of generalized eigenfunctions.**

Sum of polynomial-exponential functions

- $\mathcal{F} = \mathcal{P}ol\mathcal{E}xp(\mathbf{x})$,
- $S_j : h(x) \mapsto h(x_1, \ldots, x_{j-1}, x_j + \delta_j, x_{j+1}, \ldots, x_n)$ shift of x_j by δ_j,
- $\Delta : h(x) \mapsto \Delta[h] = h(0)$ the evaluation at 0.

Generating series of h: $\sigma_h(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} h(\alpha_1 \delta_1, \ldots, \alpha_n \delta_n) \frac{\mathbf{y}^\alpha}{\alpha!}$.
Eigenfunctions: $e^{\mathbf{f} \cdot \mathbf{x}}$; generalized eigenfunctions: $\omega(\mathbf{x}) e^{\mathbf{f} \cdot \mathbf{x}}$;

$h(\mathbf{x}) = \sum_{i=1}^{r'} g_i(\mathbf{x}) e^{\mathbf{f}_i \mathbf{x}} + r(\mathbf{x})$ with $g_i(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, $\mathbf{f}_i \in \mathbb{C}^n$ and $r(\delta \odot \alpha) = 0$, $\forall \alpha \in \mathbb{N}^n$, iff

$$\sigma_h(\mathbf{y}) = \sum_{i=1}^{r'} \omega_i(\mathbf{y}) \mathfrak{e}_{\xi_i}(\mathbf{y})$$

with $\xi_i = e^{\mathbf{f}_i} \in \mathcal{V}(\ker H_{\sigma_h}) \subset \mathbb{C}^n$, $\omega_i(\mathbf{x}) = \sum_\alpha g_{i,\alpha} \omega_\alpha$ for $g_i = \sum_\alpha g_{i,\alpha} \mathbf{x}^\alpha$.

☞ Decomposition from the moments $\sigma_\alpha = h(\alpha_1 \delta_1, \ldots, \alpha_n \delta_n)$.

Sparse interpolation of PolyLog functions

- $\mathcal{F} = \mathcal{P}oly\mathcal{L}og(\mathbf{x}) = \{\sum_{(\beta,\gamma)\in A} h_{\beta,\gamma} \log^{\beta}(\mathbf{x})\mathbf{x}^{\gamma}, A \text{ finite}\}$,
- $S_j : h(x_1, \ldots, x_n) \mapsto h(\ldots, x_{j-1}, \lambda_j x_j, x_{j+1}, \ldots)$ for $\lambda_j \in \mathbb{C} - \{1\}$,
- $\Delta : h(x_1, \ldots, x_n) \mapsto \Delta[h] = h(1, \ldots, 1)$.

Generating series of h: $\sigma_h(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} h(\lambda_1^{\alpha_1}, \ldots, \lambda_n^{\alpha_n}) \frac{\mathbf{y}^{\alpha}}{\alpha!}$.
Eigenfunctions: $\mathbf{x}^{\gamma}$; generalized eigenfunctions: $\log^{\beta}(\mathbf{x})\mathbf{x}^{\gamma}$.

$h = \sum_{i=1}^{r'} \sum_{\beta\in B_i} \omega_{i,\beta} \log^{\beta}(\mathbf{x})\,\mathbf{x}^{\gamma_i}$ iff the generating series σ_h is of the form

$$\sigma_h(\mathbf{y}) = \sum_{i=1}^{r'} \omega_i(\mathbf{y}) \mathfrak{e}_{\xi_i}(\mathbf{y})$$

with $\xi_i = (\lambda_1^{\gamma_{i,1}}, \ldots, \lambda_n^{\gamma_{i,n}}) \in \mathbb{C}^n$ and $\omega_i(\mathbf{y}) = \sum_{\beta\in B_i} \omega_{i,\beta}\mathbf{y}^{\beta} \in \mathbb{C}[\mathbf{y}]$.

☞ Decomposition from the moments $\sigma_{\alpha} = h(\lambda_1^{\alpha_1}, \ldots, \lambda_n^{\alpha_n})$.

Sparse reconstruction from Fourier coefficients

- $\mathcal{F} = L^2(\Omega)$;
- $S_i : h(x) \in L^2(\Omega) \mapsto e^{2\pi \frac{x_i}{T_i}} h(x) \in L^2(\Omega)$ is the multiplication by $e^{2\pi \frac{x_i}{T_i}}$;
- $\Delta : h(x) \in \mathcal{O}'_C \mapsto \int h(x) dx \in \mathbb{C}$.

The moments of f are

$$\sigma_\gamma = \frac{1}{\prod_{j=1}^n T_j} \int f(x) e^{-2\pi i \sum_{j=1}^n \frac{\gamma_j x_j}{T_j}} dx$$

Eigenfunctions: δ_ξ; generalized eigenfunctions: $\delta_\xi^{(\alpha)}$.
For $f \in L^2(\Omega)$ and $\sigma = (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n}$ its Fourier coefficients,

$$\Gamma_\sigma : (\rho_\beta)_{\beta \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n) \mapsto \left(\sum_\beta \sigma_{\alpha+\beta} \rho_\beta \right)_{\alpha \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n).$$

Γ_σ is of finite rank r if and only if $f = \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \omega_{i,\alpha} \delta_{\xi_i}^{(\alpha)}$ with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \Omega$, $\omega_{i,\alpha} \in \mathbb{C}$ and $r = \sum_{i=1}^{r'} \mu(\sum_{\alpha \in A_i} \omega_{i,\alpha} \mathbf{y}^\alpha)$

Other applications

- Decomposition of measures as sums of spikes from moments (images, spectroscopy, radar, astronomy, ...)
- Decomposition of convolution operators of finite rank
- Vanishing ideal of points: $\sigma = \sum_{i=1}^{r} \mathfrak{e}_{\xi_i}(\mathbf{y})$
- Change of ordering for Grobner bases or change of bases for zero-dimensional ideals: $\sigma_\alpha = \langle u, N(\mathbf{x}^\alpha) \rangle$,
- ...

Low rank decomposition of Hankel matrices

Rank 1 Hankel matrices: $H_\xi = [\xi^{\alpha+\beta}]_{\alpha \in A, \beta \in B}$ for some $\xi \in \mathbb{K}^n$ or $\mathbb{P}^n$.

Rank r Hankel matrices are not necessarily the sum of r rank one Hankel matrices.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \lambda_1 \begin{bmatrix} 1 & \xi_1 & \xi_1^2 \\ \xi_1 & \xi_1^2 & \xi_1^3 \\ \xi_1^2 & \xi_1^3 & \xi_1^4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 & \xi_2 & \xi_2^2 \\ \xi_2 & \xi_2^2 & \xi_2^3 \\ \xi_2^2 & \xi_2^3 & \xi_2^4 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \lim_{\epsilon \to 0} \frac{1}{2\,\epsilon} \begin{bmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & \epsilon^2 & \epsilon^3 \\ \epsilon^2 & \epsilon^3 & \epsilon^4 \end{bmatrix} - \frac{1}{2\,\epsilon} \begin{bmatrix} 1 & -\epsilon & \epsilon^2 \\ -\epsilon & \epsilon^2 & -\epsilon^3 \\ \epsilon^2 & -\epsilon^3 & \epsilon^4 \end{bmatrix}$$

Symbol: $y = \lim_{\epsilon \to 0} \frac{1}{2\,\epsilon}(e^{\epsilon y} - e^{-\epsilon y})$.

Structured Low rank Decomposition

Decomposition in sum of Hankel operators associated to symbols $\omega(y)\mathbf{e}_\xi(y)$ with $\omega(y) \in \mathbb{K}[\mathbf{y}], \xi \in \mathbb{C}^n$.

☞ $\sigma = \sum_{i=1}^r \omega_i \, \mathfrak{e}_{\xi_i}(\mathbf{y}) \Rightarrow$

$$H_\sigma^{A,B} = V_A(\xi_1, \dots, \xi_r)\, \Delta(\omega_1, \dots, \omega_r)\, V_B(\xi_1, \dots, \xi_r)^t$$
$$H_{g\star\sigma}^{A,B} = V_A(\xi_1, \dots, \xi_r)\, \Delta(\omega_1 g(\xi_1), \dots, \omega_r g(\xi_r))\, V_B(\xi_1, \dots, \xi_r)^t$$

where $V_A(\xi_1, \dots, \xi_r) = [\xi_j^{\alpha_i}]_{1 \le i \le |A|, 1 \le j \le r}$, $\Delta(\cdots)$ diagonal matrix.

☞ $\sigma = \sum_{i=1}^r \omega_i(\mathbf{y}) \, \mathfrak{e}_{\xi_i}(\mathbf{y}) \Rightarrow$

$$H_\sigma^{A,B} = W_{A;\mathbf{\Gamma}}(\xi) \Delta_\omega^{\mathbf{\Gamma}} W_{B;\mathbf{\Gamma}}(\xi)^t$$
$$H_{g\star\sigma}^{A,B} = W_{A;\mathbf{\Gamma}}(\xi) \Delta_{g\star\omega}^{\mathbf{\Gamma}} W_{B;\mathbf{\Gamma}}(\xi)^t$$

where $W_{A;\mathbf{\Gamma}}(\xi)$ Wronskian, $\Delta_{g\star\omega}^{\mathbf{\Gamma}}$ block diagonal.

Symmetric tensors of low rank (joint work with A. Oneto)

For $\psi \in S_d$ of degree d, with a decomposition $\psi = \sum_{i=1}^{r} \langle \xi_i, \overline{\mathbf{x}} \rangle^d$ and for $0 \le k \le d-k$,

$$H_{\psi^*}^{d-k,k} = V_{d-k}(\Xi)\, V_k^t(\Xi)$$

where $\Xi = (\xi_1, \ldots, \xi_r) \in (\mathbb{K}^{n+1})^r$, $V_k(\Xi)$ is the Vandermonde matrix of Ξ at the monomials of deg. k.

Notation

- $\psi_k^\perp = \ker H_{\psi^*}^{d-k,k}$
- $h(k) = \dim S_k/\psi_k^\perp = \mathrm{rank} H_{\psi^*}^{d-k,k}$
- $I(\Xi)$ defining ideal of the points Ξ

Apolarity lemma

Ξ is **apolar** to ψ (i.e. appears in a decomposition of ψ) iff $I(\Xi)_k \subset \psi_k^\perp$ for any $k \in \mathbb{N}$.

Proof. $\psi^* \in \langle \mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r} \rangle \subset S_d^*$.

The **regularity** of Ξ is $\rho(\Xi) = \min\{k \in \mathbb{N} \mid \exists u_1, \ldots, u_r \in S_k \text{ s.t. } u_i(\xi_j) = \delta_{i,j}\}$.

Regularity lemma

Let $\psi \in S_d$ and let Ξ be a minimal set of points apolar to ψ. Then,
$I(\Xi)_k = \psi_k^\perp \quad \text{for } 0 \leq k \leq d - \rho(\Xi)$.

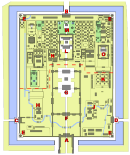
Proof. $\psi_k^\perp = \ker H_{\psi^*}^{d-k,k} = V_{d-k}(\Xi)\, V_k^t(\Xi)$, $I(\Xi)_k = \ker V_k^t(\Xi)$ and $V_{d-k}(\Xi)$ injective for $d - k \geq \rho(\Xi)$.

Theorem

Let $\psi \in S_d$ and let Ξ be a minimal set of points apolar to ψ. If $d \geq 2\rho(\Xi) + 1$, then

$$I(\Xi) = (\psi^\perp_{\leq \rho(\Xi)+1}).$$

Moreover, Ξ is the unique minimal set of points apolar to ψ.



A set of **essential variables** of ψ is a minimal set of linear forms $\ell_1, \dots, \ell_N \in S$, such that $\psi \in \mathbb{C}[\ell_1, \dots, \ell_N]$.

Proposition

- *[Car06] the number of essential variables is* $h_\psi(1)$*;*
- *[CC017] any minimal decomposition of* ψ *involves only linear forms in the essential variables.*

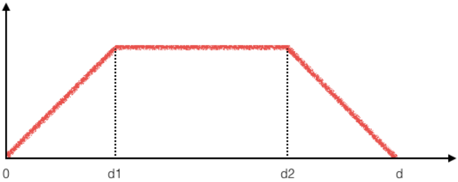
The **Waring locus** of ψ is the locus of linear forms that can appear in a minimal decomposition of ψ, i.e.,

$$\mathcal{W}_\psi := \left\{ [\ell] \in \mathbb{P}(S_1) \mid \exists \ell_2, \dots, \ell_r,\ r = \mathrm{rank}(\psi),\ \text{s.t. } \psi \in \left\langle \ell^d, \ell_2^d, \dots, \ell_r^d \right\rangle \right\};$$

The complement is **forbidden locus** denoted $\mathcal{F}_\psi := \mathbb{P}^n \setminus \mathcal{W}_\psi$.

Tensor with 2 essential variables (Sylvester method)

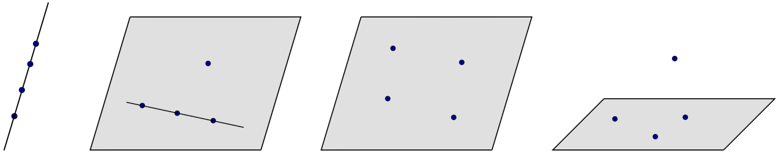
Let $\psi(x_0, x_1) \in S_d = \mathbb{K}[x_0, x_1]_d$.
The Hilbert function of $\mathcal{A}_{\psi^*}$ is of the form:



with $(\psi^\perp) = (G_1, G_2)$ of degree $0 \le d_1 \le d_2 \le d$ with $d_1 + d_2 = d + 2$.

- If G_1 has simple roots, then ψ is of rank $d_1 = \deg(G_1)$ and the roots of G_1 are the unique minimal set apolar to ψ.
- Otherwise, ψ is of rank $d_2 = \deg(G_2)$ and $\mathcal{W}_\psi$ is dense in $\mathbb{P}^1$. For a generic choice of $A \in S_{d_2 - d_1}$, the roots of $AG_1 + G_2$ are a minimal set apolar to ψ.

Cases of rank 4



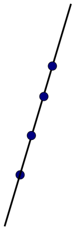
(a) Collinear

(b) Coplanar, with 3 collinear.

(c) General coplanar.

(d) General points.

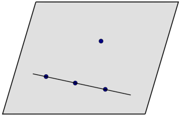
For $\psi \in S_d$ of rank 4.



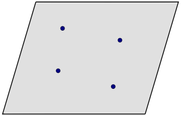
ψ **has two essential variables ($h_\psi(1) = 2$):**
$\psi^\perp = (L_1, \ldots, L_{n-1}, G_1, G_2)$, where $\deg(G_i) = d_i$ and $d_1 \leq d_2$. In particular, it has to be $d \geq 4$ and:

- **(i)** if $d = 4, 5, 6$, then $d_2 = 4$, and minimal apolar sets of points are defined by ideals $I(\Xi) = (L_1, \ldots, L_{n-1}, HG_1 + \alpha G_2)$, for a general choice of $H \in T_{6-d}$ and $\alpha \in \mathbb{C}$;
- **(ii)** if $d \geq 7$, then $d_1 = 4$ and the unique minimal apolar set of points is given by $I(\Xi) = (L_1, \ldots, L_{n-1}, G_1)$.

ψ has three essential variables ($h_\psi(1) = 3$) and a minimal apolar set Ξ of type (b):



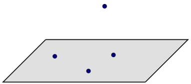
- **(i)** if $d = 3$, then $\mathcal{V}(\psi_2^\perp) = P + D$, where P is a reduced point and D is connected scheme of length 2 whose linear span is a line L_D;
  Any minimal apolar set is of the type $P \cup \Xi'$, with $\Xi' \subset L_D$;
- **(ii)** if $d = 4$, then $h_\psi(2) = 4$, $\mathcal{V}(\psi_2^\perp) = P \cup L$, where P is a reduced point and L is a line not passing through P;
  Any minimal apolar set is of the type $P \cup \Xi'$, where $\Xi' \subset L$.
- **(iii)** if $d \geq 5$, then $h_\psi(2) = 4$, $\mathcal{V}(\psi_2^\perp) = P \cup L$, where P is a reduced point and L is a line not passing through P and $(\psi_3^\perp)$ defines the unique minimal apolar set.



ψ **has three essential variables ($h_\psi(1) = 3$) and a minimal apolar set Ξ of type (c):**

(i) if $d = 3$, then $\mathcal{V}(\psi_2^\perp) = \emptyset$ and $\mathcal{W}_\psi$ is dense in the plane of essential variables;

(ii) if $d \geq 4$, there is a unique minimal apolar set of points given by $I(\Xi) = (\psi_2^\perp)$.



ψ **has four essential variables:**

there is a unique minimal apolar set of points given by $I(\Xi) = (\psi_2^\perp)$.

Classification/algorithm for rank ≤ 5

	Hilbert Sequence	Extra Condition	Algorithm to find a minimal apolar set
(1)	$[1^*]$		$\mathrm{rk}(f)=1$ *and* $(f_1^\perp)$ *defines the point apolar to* f
(2)	$[1,2,*,2,1]$		f *has two essential variables and Sylvester algorithm is applied:* (i) *if* $f^\perp_{l(f)}$ *defines a set of* $l(f)$ *reduced points, then* $\mathrm{rk}(f)=l(f)$; (ii) *otherwise,* $\mathrm{rk}(f)=d+2-l(f)$ *and a minimal apolar set is given by the principal ideal generated by a generic form* $g\in f^\perp_{d+2-l(f)}$
(3)	$[1,3,3,1]$	$Z(f_2^\perp)=\emptyset$	*a generic pair of conics* q_1,q_2 *of* $f_2^\perp$ *defines 4 points and* $\mathrm{rk}(f)=4$
(4)	$[1,3,3,1]$	$Z(f_2^\perp)=P\cup D$, P *is simple point* D *connected, 0-dim* $\deg(D)=2$	$\mathrm{rk}(f)=4$ *and* P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f'=f-c\ell_P^3$ *has two essential variables and we apply Sylvester algorithm to* f' *as in* (2)
(5)	$[1,3,3,1]$	$Z(f_2^\perp)=D$ D *connected, 0-dim* $\deg(D)=3$	$\mathrm{rk}(f)=5$ *and, for a generic* P *and a generic* $c\neq 0$ *such that* $f'=f+c\ell_P^3$ *is a ternary cubic of rank 4 and we apply* (4) *to* f'
(6)	$[1,3,3^*,3,1]$	$Z(f_2^\perp)=\{P_1,P_2,P_3\}$ P_i*'s are simple points*	$\mathrm{rk}(f)=3$ *and the unique minimal apolar set is* $Z(f_2^\perp)$
(7)	$[1,3,*,3,1]$	$Z(f_2^\perp)=P\cup L$ P *is simple point* L *is line,* $P\notin L$	P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f'=f-c\ell_P^d$ *has two essential variables and we apply Sylvester algorithm to* f' *as in* (2)
(8)	$[1,3,4^*,3,1]$	$Z(f_2^\perp)=\{P_1,\dots,P_4\}$ P_i*'s are simple points*	$\mathrm{rk}(f)=4$ *and the unique minimal apolar set is* $Z(f_2^\perp)$
(9)	$[1,3,5,3,1]$	$Z(f_2^\perp)=C$ C *is irreducible quadric*	*let* P *be a generic point on* C *and* c *be a scalar such that* $f'=f-c\ell_P^4$ *has* $h_{f'}(2)=4$. (i) *if* $Z((f')_2^\perp)=\{P_1,\dots,P_4\}$ *is a set of 4 reduced points, then,* $\mathrm{rk}(f)=5$, *and a minimal set apolar to* f *is* $\{P,P_1,\dots,P_4\}$; (ii) *otherwise,* $\mathrm{rk}(f)>5$
(10)	$[1,3,5,3,1]$	$Z(f_2^\perp)=L_1\cup L_2$ L_i*'are distinct lines*	*let* P_i *be a generic point on* L_i, *for* $i=1,2$, *respectively, and* c_i *be a scalar such that* $f_i=f-c_i\ell_{P_i}^4$ *has* $h_{f_i}(2)=4$, *for* $i=1,2$. (i) *if* $Z((f_i^\perp)_2)=\{P_1,\dots,P_4\}$, *for either* $i=1$ *or* $i=2$, *then,* $\mathrm{rk}(f)=5$, *and a minimal apolar set of* f *is* $\{P,P_1,\dots,P_4\}$; (ii) *otherwise,* $\mathrm{rk}(f)>5$
(11)	$[1,3,5,5^*,3,1]$	$Z(f_3^\perp)=\{P_1,\dots,P_5\}$ P_i*'s are reduced points*	$\mathrm{rk}(f)=5$ *and the unique minimal apolar set is* $Z(f_3^\perp)$
(12)	$[1,4,4,1]$	$Z(f_2^\perp)=P\cup H$ P *is a reduced point* H *is a plane,* $P\notin H$	P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f'=f-c\ell_P^3$ *has three essential variables and we apply* (3) *or* (4) *to* f'
(13)	$[1,4,5^*,4,1]$	$Z(f_2^\perp)=\{P_1,\dots,P_5\}$	$\mathrm{rk}(f)=5$ *and the unique minimal apolar set is* $Z(f_2^\perp)$
(14)	$[1,5,5^*,5,1]$	$Z(f_2^\perp)=\{P_1,\dots,P_5\}$	$\mathrm{rk}(f)=5$ *and the unique minimal apolar set is* $Z(f_2^\perp)$

High rank and small forbidden locus

Definition: **generic rank** $= \mathrm{rank}$ of tensors on a dense open subset of the set of tensors.

Theorem (Alexander, Hirschovitz, 1995)

The generic rank of a tensor in $\mathbb{K}[x_0, \ldots, x_n]_d$ *is* $\lceil \frac{1}{n+1}\binom{n+d}{d} \rceil$, *except for* $d = 2$ *and* $(n, d) \in \{(2, 4), (3, 4), (4, 3), (4, 4)\}$.

Theorem (Oneto,·, 2018)

Let g *be the generic rank of tensors of degree* d *in* $\mathbb{P}^n$. *Let* $\psi \in S_d$ *with* $r = \mathrm{rank}(\psi)$. *If* $r > g$, *then* $\mathcal{W}_\psi$ *is dense in* $\mathbb{P}^n$.

Flat extension of a truncated moment matrix

For (monomial) sets $B \subset C$, $B' \subset C'$, $\overline{B} = C \setminus B$, $\overline{B}' = C' \setminus B'$.

$$\mathbf{H}_\sigma^{\mathbf{C},\mathbf{C}'} = \left(\langle \sigma \mid \mathbf{x}^{\alpha+\beta} \rangle\right)_{\alpha \in C, \beta \in C'} = \left[\begin{array}{c|c} \mathbf{H}_\sigma^{\mathbf{B},\mathbf{B}'} & \mathbf{H}_\sigma^{\mathbf{B},\overline{\mathbf{B}}'} \\ \hline \mathbf{H}_\sigma^{\overline{\mathbf{B}},\mathbf{B}'} & \mathbf{H}_\sigma^{\overline{\mathbf{B}},\overline{\mathbf{B}}'} \end{array}\right],$$

when

$$\mathrm{rank}\, \mathbf{H}_\sigma^{\mathbf{C},\mathbf{C}'} = \mathrm{rank}\, \mathbf{H}_\sigma^{\mathbf{B},\mathbf{B}'}$$

For $B \subset \mathbb{K}[\mathbf{x}]$, let $B^+ = B \cup x_1 B \cdots x_n B$, $\partial B = B^+ \setminus B$.

Theorem

Assume $H_\sigma^{B,B'}$ invertible with $|B| = |B'| = r$ and $C \supset B^+, C' \supset B'^+$ connected to 1 ($m \in C \Rightarrow m = 1$ or $m = x_j m'$ with $m' \in C$).

$H_\sigma^{C,C'}$ is a **flat extension** *of $H_\sigma^{B,B'}$*

$\Leftrightarrow$ *The operators $M_j := H_\sigma^{B,x_jB}(H_\sigma^{B,B})^{-1}$* **commute**.

$\Leftrightarrow$ $\exists! \, \tilde{\sigma} \in \mathcal{P}\mathbf{ol}\mathcal{E}\mathbf{xp}$ *s.t.* $\mathrm{rank} H_{\tilde{\sigma}} = r$ *and* $\tilde{\sigma}_{|C \cdot C'} = \sigma$.

Example

$$\begin{aligned}\sigma \quad = \quad & 8 + 17\, z_2 - 4\, z_1 + 15\, z_2^2 + 14\, z_1 z_2 - 16\, z_1^2 + 47\, z_2^3 - 6\, z_1 z_2^2 \\ & + 34\, z_1^2 z_2 - 52\, z_1^3 + 51\, z_2^4 + 38\, z_1 z_2^3 - 18\, z_1^2 z_2^2 + 86\, z_1^3 z_2 - 160\, z_1^4\end{aligned}$$

moment series $\in \mathbb{K}[[z_1, z_2]]$, truncated in degree 4.

$$[H_\sigma^{B^+,B^+}] = \begin{bmatrix} 8 & -4 & 17 & -16 & 14 & 15 & -52 & 34 & -6 & 47 \\ -4 & -16 & 14 & -52 & 34 & -6 & -160 & 86 & -18 & 38 \\ 17 & 14 & 15 & 34 & -6 & 47 & 86 & -18 & 38 & 51 \\ -16 & -52 & 34 & -160 & 86 & -18 & h_1 & h_2 & h_3 & h_4 \\ 14 & 34 & -6 & 86 & -18 & 38 & h_2 & h_3 & h_4 & h_5 \\ 15 & -6 & 47 & -18 & 38 & 51 & h_3 & h_4 & h_5 & h_6 \\ -52 & -160 & 86 & h_1 & h_2 & h_3 & h_7 & h_8 & h_9 & h_{10} \\ 34 & 86 & -18 & h_2 & h_3 & h_4 & h_8 & h_9 & h_{10} & h_{11} \\ -6 & -18 & 38 & h_3 & h_4 & h_5 & h_9 & h_{10} & h_{11} & h_{12} \\ 47 & 38 & 51 & h_4 & h_5 & h_6 & h_{10} & h_{11} & h_{12} & h_{13} \end{bmatrix}$$

where $B = \{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$.

Flat extension condition: $\mathrm{rank} H_\sigma^{B^+,B^+} \leq 6$ implies

$$\left\{\begin{array}{l}
-814592\, h_1^2 - 1351680\, h_1 h_2 - 476864\, h_1 h_3 - 599040\, h_2^2 - 301440\, h_2 h_3 - 35072\, h_3^2 \\
\quad -520892032\, h_1 - 396821760\, h_2 - 164529152\, h_3 + 1693440\, h_7 - 86394672128 = 0 \\
-814592\, h_2^2 - 1351680\, h_2 h_3 - 476864\, h_2 h_4 - 599040\, h_3^2 - 301440\, h_3 h_4 - 35072\, h_4^2 \\
\quad +335275392\, h_2 + 257276160\, h_3 + 96277632\, h_4 + 1693440\, h_9 - 34904464128 = 0 \\
\quad \vdots \\
-814592\, h_1 h_3 - 675840\, h_1 h_4 - 238432\, h_1 h_5 - 675840\, h_2 h_3 - 599040\, h_2 h_4 - 150720\, h_2 h_5 - 238432\, h_3^2 \\
\quad -150720\, h_3 h_4 - 35072\, h_3 h_5 + 6613440\, h_1 + 6641280\, h_2 - 264559616\, h_3 - 198410880\, h_4 - 82264576\, h_5 \\
\quad +1693440\, h_9 + 1312368000 = 0 \\
-814592\, h_1 h_4 - 675840\, h_1 h_5 - 238432\, h_1 h_6 - 675840\, h_2 h_4 - 599040\, h_2 h_5 - 150720\, h_2 h_6 - 238432\, h_3 h_4 \\
\quad -150720\, h_3 h_5 - 35072\, h_3 h_6 + 106430368\, h_1 + 81349440\, h_2 + 25713728\, h_3 - 260446016\, h_4 \\
\quad -198410880\, h_5 - 82264576\, h_6 + 1693440\, h_{10} + 34550702464 = 0
\end{array}\right.$$

Solution set: an algebraic variety of dimension 3 and degree 52.

A solution (among others) is $h_1 = -484,\ h_2 = 226,\ h_3 = -54,\ h_4 = 82,\ h_5 = -6,\ h_6 = 167,$ $h_7 = -1456,\ h_8 = 614,\ h_9 = -162,\ h_{10} = 182,\ h_{11} = -18,\ h_{12} = 134,\ h_{13} = 195.$

Decomposition of **rank 6** of the series with these computed moments:

$$\begin{aligned}
\sigma \quad \equiv \quad & (0.517 + 0.044\, i)\, \mathbf{e}_{-0.830+1.593\, i,\, -0.326-0.050\, i} + (0.517 - 0.044\, i)\, \mathbf{e}_{-0.830-1.593\, i,\, -0.326+0.050\, i} \\
& +2.958\, \mathbf{e}_{1.142,\, 0.836} + 4.583\, \mathbf{e}_{0.956,\, -0.713} \\
& -(4.288 + 1.119\, i)\, \mathbf{e}_{-0.838+0.130\, i,\, 0.060+0.736\, i} - (4.288 - 1.119\, i)\, \mathbf{e}_{-0.838-0.130\, i,\, 0.060-0.736\, i}
\end{aligned}$$

General decomposition algorithm [BCMT10], [BS18]

- Perform a generic change of coordinates $\psi'(\bar{\mathbf{x}}) = \psi(T\,\bar{\mathbf{x}})$.
- For $r = \max \mathrm{rank} H_\sigma^{k,d-k}, \ldots$
  - For bases B, B' of size r, connected to 1 (e.g. B stable by division/Borel fixed stable by division);
    1. **Find the (unknown) moments** of $H_\Lambda^{B'^+,B^+}$ s.t.
       - $H_\Lambda^{B',B}$ invertible and
       - the operators $M_i = H_\Lambda^{x_i B',B}(H_\Lambda^{B',B})^{-1}$ commute.
    2. Deduce the decomposition of σ (Algorithm 1).
    3. If the roots are simple and the decomposition is valid for the moments of ψ, **stop and output a decomposition of ψ**;

Challenges, open questions

- Numerical stability, correction of errors,
- Efficient construction of basis, complexity,
- Super-resolution, collision of points,
- Super-extrapolation,
- Best low rank approximation,
- ...

Thanks for your attention

References

Alessandra Bernardi, Jérôme Brachat, Pierre Comon, and Bernard Mourrain.
General tensor decomposition, moment matrices and applications.
Journal of Symbolic Computation, 52:51–71, 2013.

Jérôme Brachat, Pierre Comon, Bernard Mourrain, and Elias P. Tsigaridas.
Symmetric tensor decomposition.
Linear Algebra and Applications, 433(11-12):1851–1872, 2010.

Enrico Carlini.
Reducing the number of variables of a polynomial.
In *Algebraic geometry and geometric modeling*, pages 237–247. Springer, 2006.

Enrico Carlini, Maria Virginia Catalisano, and Alessandro Oneto.
Waring loci and the Strassen conjecture.
Advances in Mathematics, 314:630–662, 2017.

Cédric Josz, Jean Bernard Lasserre, and Bernard Mourrain.
Sparse polynomial interpolation: Compressed sensing, super resolution, or Prony?
hal-01575325, ArXiv:1708.06187, August 2017.

Monique Laurent and Bernard Mourrain.
A generalized flat extension theorem for moment matrices.
Archiv der Mathematik, 93(1):87–98, 2009.

Bernard Mourrain.
Polynomial-Exponential Decomposition from Moments.
Foundations of Computational Mathematics, 2017.
http://dx.doi.org/10.1007/s10208-017-9372-x.

Bernard Mourrain and Alessandro Oneto.
On minimal decompositions of low rank symmetric tensors.
preprint, https://hal.archives-ouvertes.fr/hal-01803571, May 2018.