Varieties of Signature Tensors

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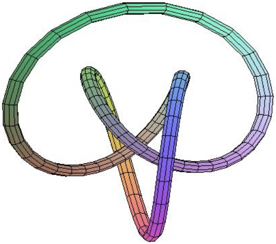

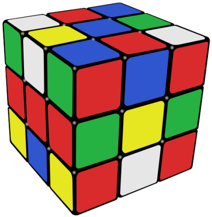

Work with Carlos Améndola and Peter Friz and related articles

Paths

A *path* is a piecewise differentiable map $X : [0,1] \to \mathbb{R}^d$.

Coordinate functions: $X_1, X_2, \ldots, X_d \; : \; \mathbb{R} \to \mathbb{R}$

Their differentials

$$\mathrm{d}X_i(t) \;=\; X_i'(t)\mathrm{d}t$$

are the coordinates of the vector

$$\mathrm{d}X \;=\; \big(\mathrm{d}X_1, \mathrm{d}X_2, \ldots, \mathrm{d}X_d\big).$$

Fundamental Theorem of Calculus:

$$\int_0^1 \mathrm{d}X_i(t) \;=\; X_i(1) - X_i(0)$$

The *first signature* of the path X is

$$\sigma^{(1)}(X) \;=\; \int_0^1 \mathrm{d}X(t) \;=\; X(1) - X(0) \;\in\; \mathbb{R}^d.$$

Signature Matrices

Fix a path $X : [0,1] \to \mathbb{R}^d$ with $X(0) = 0$.

Its *second signature* $S = \sigma^{(2)}(X)$ is the $d \times d$ matrix with entries

$$\sigma_{ij} = \int_0^1 \int_0^t \mathrm{d}X_i(s)\, \mathrm{d}X_j(t).$$

By the Fundamental Theorem of Calculus,

$$\sigma_{ij} = \int_0^1 X_i(t) X_j'(t) \mathrm{d}t.$$

The symmetric matrix $S + S^T$ has rank one. Its entries are

$$\sigma_{ij} + \sigma_{ji} = X_i(1) \cdot X_j(1).$$

In matrix notation,

$$S + S^T = X(1)^T X(1).$$

The skew-symmetric matrix $S - S^T$ measures *deviation from linearity*:

$$\sigma_{ij} - \sigma_{ji} = \int_0^1 \big(X_i(t) X_j'(t) - X_j(t) X_i'(t)\big) \mathrm{d}t$$

Lévy Area

The entry $\sigma_{ij} - \sigma_{ji}$ of the skew-symmetric matrix $S - S^T$ is the area below the line minus the area above the line:

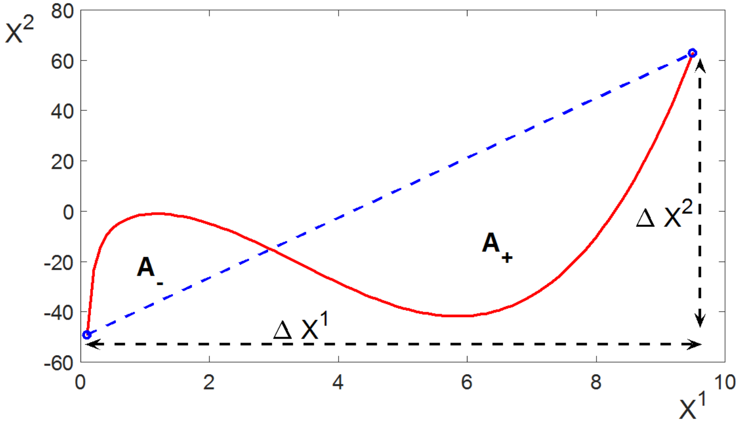

Figure 5: Example of signed Lévy area of a curve. Areas above and under the chord connecting two endpoints are negative and positive respectively.

Signature Tensors

The *kth signature* of X is a tensor $\sigma^{(k)}(X)$ of order k and format $d \times d \times \cdots \times d$. Its d^k entries $\sigma_{i_1 i_2 \cdots i_k}$ are the *iterated integrals*

$$\sigma_{i_1 i_2 \cdots i_k} = \int_0^1 \int_0^{t_k} \cdots \int_0^{t_3} \int_0^{t_2} \mathrm{d}X_{i_1}(t_1)\, \mathrm{d}X_{i_2}(t_2) \cdots \mathrm{d}X_{i_{k-1}}(t_{k-1})\, \mathrm{d}X_{i_k}(t_k).$$

The tensor equals

$$\sigma^{(k)}(X) = \int_\Delta \mathrm{d}X(t_1) \otimes \mathrm{d}X(t_2) \otimes \cdots \otimes \mathrm{d}X(t_k),$$

where the integral is over the simplex

$$\Delta = \{ (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1 \}.$$

We are interested in projective varieties in tensor space $\mathbb{P}^{d^k - 1}$ that arise when X ranges over some nice families of paths.

Example: For linear paths X, we get the Veronese variety.

Planar Example

Consider quadratic paths in the plane $\mathbb{R}^2$:

$$X(t) \;=\; (x_{11}t + x_{12}t^2, x_{21}t + x_{22}t^2)^T \;=\; \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

The third signature $\sigma^{(3)}(X)$ is a $2{\times}2{\times}2$ tensor. Its entries are

$$\begin{aligned}
\sigma_{111} &= \tfrac{1}{6}(x_{11}+x_{12})^3 \\
\sigma_{112} &= \tfrac{1}{6}(x_{11}+x_{12})^2(x_{21}+x_{22}) + \tfrac{1}{60}(5x_{11}+4x_{12})(x_{11}x_{22}-x_{21}x_{12}) \\
\sigma_{121} &= \tfrac{1}{6}(x_{11}+x_{12})^2(x_{21}+x_{22}) + \tfrac{1}{60}(2x_{12})(x_{11}x_{22}-x_{21}x_{12}) \\
\sigma_{211} &= \tfrac{1}{6}(x_{11}+x_{12})^2(x_{21}+x_{22}) - \tfrac{1}{60}(5x_{11}+6x_{12})(x_{11}x_{22}-x_{21}x_{12}) \\
\sigma_{122} &= \tfrac{1}{6}(x_{11}+x_{12})(x_{21}+x_{22})^2 + \tfrac{1}{60}(5x_{21}+6x_{22})(x_{11}x_{22}-x_{21}x_{12}) \\
\sigma_{212} &= \tfrac{1}{6}(x_{11}+x_{12})(x_{21}+x_{22})^2 - \tfrac{1}{60}(2x_{22})(x_{11}x_{22}-x_{21}x_{12}) \\
\sigma_{221} &= \tfrac{1}{6}(x_{11}+x_{12})(x_{21}+x_{22})^2 - \tfrac{1}{60}(5x_{21}+4x_{22})(x_{11}x_{22}-x_{21}x_{12}) \\
\sigma_{222} &= \tfrac{1}{6}(x_{21}+x_{22})^3
\end{aligned}$$

This defines a threefold of degree 6 in $\mathbb{P}^7$, cut out by 9 quadrics.

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My Favorite Tensors

Example (The Canonical Axis Path)

Let C_{axis} be the path from $(0, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$ given by d linear steps in unit directions $e_1, e_2, \ldots, e_d$. The entry $\sigma_{i_1 i_2 \cdots i_k}$ of the signature tensor $\sigma^{(k)}(C_{\text{axis}})$ is zero unless $i_1 \leq i_2 \leq \cdots \leq i_k$.

In that case, it equals $1/k!$ times the number of distinct permutations of the string $i_1 i_2 \cdots i_k$. For example, if $k = 4$ then $\sigma_{1111} = \frac{1}{24}$, $\sigma_{1112} = \frac{1}{6}$, $\sigma_{1122} = \frac{1}{4}$, $\sigma_{1123} = \frac{1}{2}$, $\sigma_{1234} = 1$ and $\sigma_{1243} = 0$.

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Example (The Canonical Monomial Path)

Let C_{mono} be the monomial path $t \mapsto (t, t^2, t^3, \ldots, t^d)$. It travels from $(0, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$ along the rational normal curve. Entries of the signature tensor $\sigma^{(k)}(C_{\text{mono}})$ are

$$\sigma_{i_1 i_2 \cdots i_k} = \frac{i_1}{i_1} \cdot \frac{i_2}{i_1 + i_2} \cdot \frac{i_3}{i_1 + i_2 + i_3} \cdots \frac{i_k}{i_1 + i_2 + \cdots + i_k}.$$

My Favorite Matrices

The signature matrices of the two canonical paths are

$$\sigma^{(2)}(C_{\mathrm{axis}}) = \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \sigma^{(2)}(C_{\mathrm{mono}}) = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{2}{4} & \frac{3}{5} \\ \frac{1}{4} & \frac{2}{5} & \frac{3}{6} \end{pmatrix}.$$

The symmetric part of each matrix is the same constant rank 1 matrix:

$$\sigma^{(2)}(C_\bullet) + \sigma^{(2)}(C_\bullet)^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We encode cubic paths and three-segment paths by 3×3 matrices

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

The map $X \mapsto \sigma^{(2)}(X)$ from paths to signature matrices
is given by the congruence action $\mathbf{X} \mapsto \mathbf{X} \cdot \sigma^{(2)}(C_\bullet) \cdot \mathbf{X}^{\mathbf{T}}$.

The Skyline Path

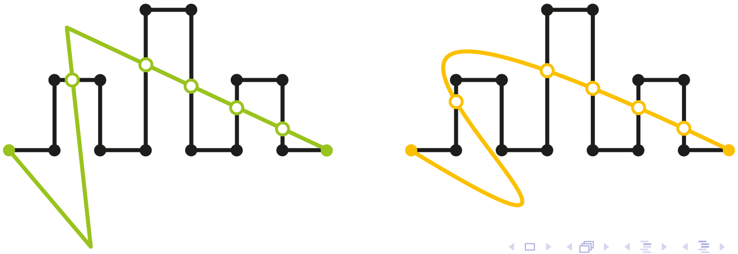
is the following axis path with 13 steps in $\mathbb{R}^2$:

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 2 & 0 & -2 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

Its $2\times2\times2$ signature tensor can be gotten from the core tensor C_{axis} of size $13\times13\times13$ by multiplying with the 2×13 matrix X on all three sides:

$$S_{\mathrm{skyline}} = [\![C_{\mathrm{axis}}; X, X, X]\!] = \frac{1}{6}\left[\begin{array}{cc|cc} 343 & 0 & -84 & 18 \\ 84 & 18 & -36 & 0 \end{array}\right].$$

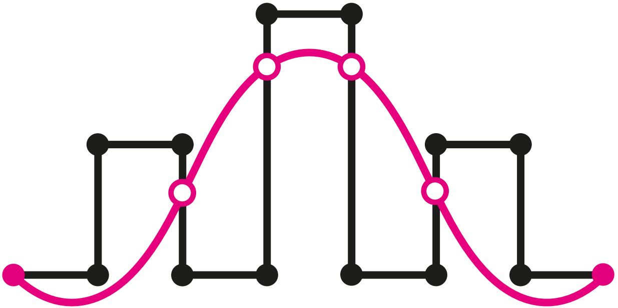
Three-step path and cubic path with the same signature tensor:



Shortest Path

... for a given signature tensor

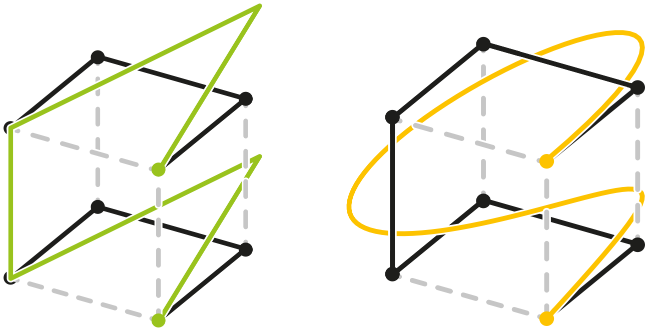
$$\sigma^{(3)}(X) \;=\; \left[\begin{array}{cc|cc} 343 & 0 & -84 & 18 \\ 84 & 18 & -36 & 0 \end{array}\right].$$



[M. Pfeffer, A. Seigal, B.St: *Learning Paths from Signature Tensors*]

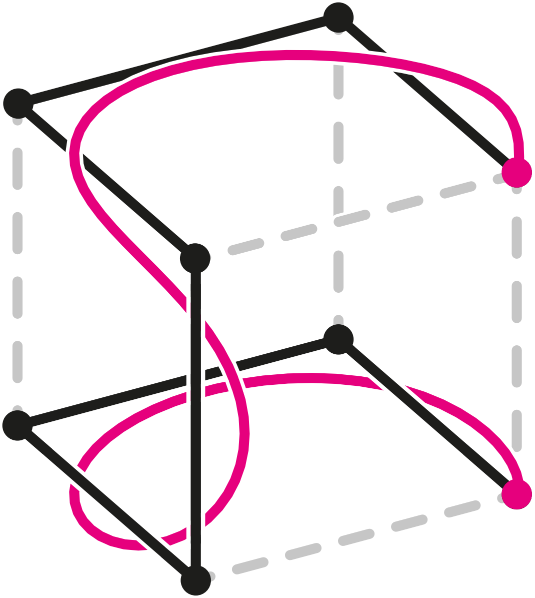
Klee-Minty Path

$$X = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$



$$\sigma^{(3)}(X) = \frac{1}{6}\left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & -6 & 3 & 3 \\ 0 & 6 & 0 & -6 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

Shortest Path



Pop Quiz

Fix $d = 2$ and consider the parametrization of the unit circle

$$X : [0,1] \to \mathbb{R}^2 , \; t \mapsto \big(\sin(2\pi t), \cos(2\pi t) \big).$$

- Compute the signature vector $\sigma^{(1)}(X)$.
- Compute the signature matrix $\sigma^{(2)}(X)$.

Yes, you can do this !!!

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Answer: $\sigma^{(3)}(X) = -\pi(e_{122} - 2e_{212} + e_{221})$.

Inverse Problem:
To what extent is a path determined by its signature tensors?

Signature Matrices

Theorem

Let $k = 2$ and $m \leq d$. Our two favorite $m \times m$ matrices $\sigma^{(2)}(C_{\mathrm{axis}})$ and $\sigma^{(2)}(C_{\mathrm{mono}})$ (padded by zeros) lie in the same orbit for the action of $\mathrm{GL}_d(\mathbb{R})$ by congruence on $d \times d$ matrices.

The orbit closure is the signature variety $\mathcal{M}_{d,m}$ in $\mathbb{P}^{d^2-1}$.

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The orbit closure is the signature variety $\mathcal{M}_{d,m}$ in $\mathbb{P}^{d^2-1}$.

Any $d \times d$ matrix $S = \sigma^{(2)}(X)$ is uniquely the sum of a symmetric matrix and a skew-symmetric matrix:

$$S \;=\; P + Q, \quad \text{where} \quad P = \frac{1}{2}(S + S^T) \quad \text{and} \quad Q = \frac{1}{2}(S - S^T).$$

The $\binom{d+1}{2}$ entries p_{ij} of P and $\binom{d}{2}$ entries q_{ij} of Q serve as coordinates on the space $\mathbb{P}^{d^2-1}$ of matrices $S = (\sigma_{ij})$.

Determinantal Varieties

Theorem

For each d and m, the following subvarieties of $\mathbb{P}^{d^2-1}$ coincide:

1. *Signature matrices of piecewise linear paths with m segments.*
2. *Signature matrices of polynomial paths of degree m.*
3. *Matrices $P+Q$, with P symmetric, Q skew-symmetric, such that*

$$\operatorname{rank}(P) \leq 1 \quad \text{and} \quad \operatorname{rank}\big([\,P\,Q\,]\big) \leq m.$$

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For each fixed d, these varieties $\mathcal{M}_{d,m}$ form a nested family:

$$\mathcal{M}_{d,1} \subset \mathcal{M}_{d,2} \subset \mathcal{M}_{d,3} \subset \cdots \subset \mathcal{M}_{d,d} = \mathcal{M}_{d,d+1} = \cdots$$

Fix $m \leq d$. Then $\mathcal{M}_{d,m}$ is irreducible of dimension $md - \binom{m}{2} - 1$ and has singular locus $\mathcal{M}_{d,m-1}$. For m odd, its ideal is generated by the 2-minors of P and $(m+1)$-pfaffians of Q. For m even, take the 2-minors of P, $(m+2)$-pfaffians of Q, and entries in $P \cdot C_m(Q)$ where $C_m(Q)$ is the circuit matrix formed by the m-pfaffians.

Example: Quadratic Paths in 3-Space

The variety $\mathcal{M}_{3,2}$ has dimension 4 and degree 6 in $\mathbb{P}^8$. It is the Zariski closure of the common $\mathrm{GL}(3)$-orbit of the two matrices

$$\sigma^{(2)}(C_{\mathrm{axis}}) = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \sigma^{(2)}(C_{\mathrm{mono}}) = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is cut out by the 2-minors of $P = (p_{ij})$ and the 3-minors of

$$\begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & 0 & q_{12} & q_{13} \\ p_{12} & p_{22} & p_{23} & -q_{12} & 0 & q_{23} \\ p_{13} & p_{23} & p_{33} & -q_{13} & -q_{23} & 0 \end{bmatrix}.$$

These do not generate the prime ideal of $\mathcal{M}_{3,2}$.
We also need the entries of $P \cdot C_2(Q)$ where $C_2(Q) = \left[q_{23}, -q_{13}, q_{12}\right]^T$.

The universal variety $\mathcal{U}_{3,2} = \mathcal{M}_{3,3} \subset \mathbb{P}^8$ is
a cone over the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

Universal Varieties

The kth signature tensor of a path X in $\mathbb{R}^d$ is a point $\sigma^{(k)}(X)$ in the tensor space $(\mathbb{R}^d)^{\otimes k}$, and hence in the projective space $\mathbb{P}^{d^k-1}$.

Consider the set of signature tensors $\sigma^{(k)}(X)$, as X ranges over **all** paths $[0,1] \to \mathbb{R}^d$. This is the universal variety $\mathcal{U}_{d,k} \subset \mathbb{P}^{d^k-1}$.

$d \setminus k$	2	3	4	5	6	7	8	9
2	2	4	7	13	22	40	70	126
3	**5**	13	31	79	195	507	1317	3501
4	9	29	89	293	963	3303	11463	40583
5	14	54	204	828	3408	14568	63318	280318

Table: The dimension of $\mathcal{U}_{d,k}$ is much smaller than $d^k - 1$.

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Table: The dimension of $\mathcal{U}_{d,k}$ is much smaller than $d^k - 1$.

Theorem

The dimension of the universal variety $\mathcal{U}_{d,k}$ is the number of Lyndon words of length $\leq k$ over the alphabet $\{1, 2, \ldots, d\}$.

A word is a *Lyndon word* if it is strictly smaller in lexicographic order than all of its rotations.

Tensors

The *truncated tensor algebra* is a non-commutative algebra:

$$T^n(\mathbb{R}^d) \;=\; \bigoplus_{k=0}^{n} (\mathbb{R}^d)^{\otimes k}$$

Standard basis given by words of length $\leq n$ on $\{1, 2, \ldots, d\}$:

$$e_{i_1 i_2 \cdots i_k} \;:=\; e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \quad \text{for } 1 \leq i_1, \ldots, i_k \leq d \text{ and } 0 \leq k \leq n.$$

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$T^n(\mathbb{R}^d)$ is a commutative algebra with respect to the *shuffle product* $\sqcup\!\sqcup$. The shuffle product of two words of lengths r and s is the sum over all $\binom{r+s}{s}$ ways of interleaving the two words:

$$e_{12} \sqcup\!\sqcup e_{34} = e_{12 \sqcup\!\sqcup 34} = e_{1234} + e_{1324} + e_{1342} + e_{3124} + e_{3142} + e_{3412}$$

$$e_{3 \sqcup\!\sqcup 134} = e_{3134} + 2e_{1334} + e_{1343} \qquad e_{21 \sqcup\!\sqcup 21} = 2e_{2121} + 4e_{2211}$$

Free Lie Algebra

$\mathrm{Lie}^n(\mathbb{R}^d)$ is the smallest Lie subalgebra of $T^n(\mathbb{R}^d)$ containing $\mathbb{R}^d$.
This is a linear subspace of $T_0^n(\mathbb{R}^d) = \{0\} \oplus \mathbb{R}^d \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes n}$.

Lemma

This is characterized by the vanishing of all shuffle linear forms:

$$\mathrm{Lie}^n(\mathbb{R}^d) \; = \; \left\{ P \in T_0^n(\mathbb{R}^d) \; : \; \sigma_{I \sqcup\!\sqcup J}(P) = 0 \; \text{ for all words } I, J \right\}.$$

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Theorem

Basis for $\mathrm{Lie}^n(\mathbb{R}^d)$ *is given by Lie bracketings of all Lyndon words.*

[C. Reutenauer: *Free Lie Algebras*, Oxford University Press, 1993]

Example. $\mathrm{Lie}^4(\mathbb{R}^2)$ is 8-dimensional in $T_0^4(\mathbb{R}^2) \simeq \mathbb{R}^{30}$. The eight Lyndon words $1, 2, 12, 112, 122, 1112, 1122, 1222$ determine a basis:

$$e_1, e_2, [e_1, e_2] = e_{12} - e_{21}, \ldots, [[[e_1, e_2], e_2], e_2] = e_{1222} - 3e_{2122} + 3e_{2212} - e_{2221}$$

The 22-dim'l space of linear relations is spanned by shuffles

$$\sigma_{21 \sqcup\!\sqcup 21} = 2\sigma_{2121} + 4\sigma_{2211}\,, \quad \sigma_{1 \sqcup\!\sqcup 111} = 4\sigma_{1111},$$
$$\sigma_{12 \sqcup\!\sqcup 21} \;=\; 2\sigma_{1221} + \sigma_{1212} + \sigma_{2121} + 2\sigma_{2112}.$$

Free Lie Group

The following are polynomial maps on $T_0^n(\mathbb{R}^d)$:

$$\exp(P) \;=\; \sum_{r\geq 0} \frac{1}{r!} P^{\otimes r} \quad \text{and} \quad \log(1+P) \;=\; \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} P^{\otimes r}.$$

The logarithm inverts the exponential function:

$$\log(\exp(P)) \;=\; P \qquad \text{for all } P \in T_0^n(\mathbb{R}^d).$$

The *step-n free Lie group* is the image of the free Lie algebra:

$$\mathcal{G}^n(\mathbb{R}^d) \;:=\; \exp(\mathrm{Lie}^n(\mathbb{R}^d)).$$

Theorem

This Lie group is an algebraic variety in $T_1^n(\mathbb{R}^d)$. Its is defined by

$$\sigma_{I \sqcup\!\sqcup J}(P) = \sigma_I(P)\sigma_J(P) \quad \textit{for all words } I, J \textit{ with } |I| + |J| \leq n.$$

Our contribution: This is the prime ideal. We have a nice Gröbner basis.

Example

The Lie algebra $\mathrm{Lie}^3(\mathbb{R}^2)$ has dimension 5:

$$\sigma \;=\; re_1 + se_2 + t[e_1,e_2] + u[e_1,[e_1,e_2]] + v[[e_1,e_2],e_2], \qquad r,s,t,v,u \in \mathbb{R}.$$

The exponential map from $\mathrm{Lie}^3(\mathbb{R}^2)$ into $T_1^3(\mathbb{R}^2) \simeq \mathbb{R}^{14}$ is

$$\exp(\sigma) \;=\; 1 + re_1 + se_2 + \frac{r^2}{2}e_{11} + \big(\frac{rs}{2}+t\big)e_{12} + \big(\frac{rs}{2}-t\big)e_{21} + \cdots$$

$$\cdots \;+\; \big(\frac{rs^2}{6}-2v\big)e_{212} + \big(\frac{rs^2}{6}-\frac{st}{2}+v\big)e_{221} + \frac{s^3}{6}e_{222}.$$

Its image is the 5-dimensional Lie group $\mathcal{G}_{2,3}$, defined by

$$\begin{gathered}\big\langle \sigma_1^2 - 2\sigma_{11}\,,\ \sigma_1\sigma_2 - \sigma_{12} - \sigma_{21}\,,\ \sigma_1\sigma_2 - \sigma_{12} - \sigma_{21}\,,\ \sigma_2^2 - 2\sigma_{22},\\ \sigma_1\sigma_{11} - 3\sigma_{111}, \sigma_1\sigma_{12} - 2\sigma_{112} - \sigma_{121}\,,\ \sigma_1\sigma_{21} - \sigma_{121} - 2\sigma_{211},\\ \sigma_1\sigma_{22} - \sigma_{122} - \sigma_{212} - \sigma_{221}\,,\ \sigma_2\sigma_{11} - \sigma_{121} - \sigma_{211} - \sigma_{112},\\ \sigma_2\sigma_{12} - 2\sigma_{122} - \sigma_{212}\,,\ \sigma_2\sigma_{21} - 2\sigma_{221} - \sigma_{212}\,,\ \sigma_2\sigma_{22} - 3\sigma_{222}\big\rangle\end{gathered}$$

Back to Paths

The connection to paths comes from the following key result.
This is attributed to Chow (1940) and Chen (1957).

Theorem (Chen-Chow)

The step-n free nilpotent Lie group $\mathcal{G}^n(\mathbb{R}^d)$ is precisely the image of the step n signature map applied to all paths in $\mathbb{R}^d$:

$$\mathcal{G}^n(\mathbb{R}^d) \;=\; \left\{ \sigma^{\leq n}(X) \;:\; X : [0,1] \to \mathbb{R}^d \text{ any smooth path} \right\}$$

Let X be the piecewise linear path with steps $X_1, X_2, \ldots, X_m$ in $\mathbb{R}^d$. Chen (1954) showed that the n-step signature of the path X is given by the tensor product of tensor exponentials:

$$\sigma^{\leq n}(X) \;=\; \exp(X_1) \otimes \exp(X_2) \otimes \cdots \otimes \exp(X_m) \quad \in \; T^n(\mathbb{R}^d).$$

The Universal Variety

We focus on signature tensors $\sigma^{(k)}(X)$ of a fixed order k.
Consider the projection of the free Lie group $\mathcal{G}_{d,k}$ into $(\mathbb{R}^d)^{\otimes k}$.

The image is an affine cone. The corresponding projective variety in $\mathbb{P}^{d^k-1}$ is denoted $\mathcal{U}_{d,k}$ and is called the *universal variety*.

Corollary

The universal variety $\mathcal{U}_{d,k}$ is the projective variety given by the kth signature tensors $\sigma^{(k)}(X)$ of all paths X in $\mathbb{R}^d$.

Example ($k = 2$)

The universal variety $\mathcal{U}_{d,2}$ of signature matrices consists of all $d \times d$ matrices whose symmetric part has rank 1.

Example ($d = 2, k = 3$)

The universal variety $\mathcal{U}_{2,3}$ for $2 \times 2 \times 2$ tensors has dimension 4 and degree 4 in $\mathbb{P}^7$. Its singular locus is a line. Equations? Geometry?

Census

With Améndola and Friz, we conjectured that the prime ideal of the universal variety $\mathcal{U}_{d,k}$ is always generated by quadrics:

d	k	amb	dim	deg	gens
2	3	7	4	4	6
2	4	15	7	12	33
2	5	31	13	40	150
3	3	26	13	24	81
3	4	80	31	672	954
4	3	63	29	200	486

Table: The prime ideals of the universal varieties $\mathcal{U}_{d,k}$

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Francesco Galuppi found a change of coordinates for $k \leq 3$ which turns $\mathcal{U}_{d,k}$ into a projective toric variety. Using these coordinates, he was able to disprove our conjecture.

Exercises

- Start with the ideal of the Lie group $\mathcal{G}_{2,3}$:

$$\begin{gathered}\langle \sigma_1^2 - 2\sigma_{11}\,,\, \sigma_1\sigma_2 - \sigma_{12} - \sigma_{21}\,,\, \sigma_1\sigma_2 - \sigma_{12} - \sigma_{21}\,,\, \sigma_2^2 - 2\sigma_{22}, \\ \sigma_1\sigma_{11} - 3\sigma_{111}, \sigma_1\sigma_{12} - 2\sigma_{112} - \sigma_{121}\,,\, \sigma_1\sigma_{21} - \sigma_{121} - 2\sigma_{211}, \\ \sigma_1\sigma_{22} - \sigma_{122} - \sigma_{212} - \sigma_{221}\,,\, \sigma_2\sigma_{11} - \sigma_{121} - \sigma_{211} - \sigma_{112}\,, \\ \sigma_2\sigma_{12} - 2\sigma_{122} - \sigma_{212}\,,\, \sigma_2\sigma_{21} - 2\sigma_{221} - \sigma_{212}\,,\, \sigma_2\sigma_{22} - 3\sigma_{222} \rangle\end{gathered}$$

  Eliminate the six unknowns $\sigma_1, \sigma_2, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ to get the ideal of the universal variety $\mathcal{U}_{2,3} \subset \mathbb{P}^7$. What is this variety?

- The Lie group $\mathcal{G}_{3,3}$ is an affine variety in $T_1^3(\mathbb{R}^3) \simeq \mathbb{R}^{39}$. Find a Gröbner basis for its ideal. What is the dimension of $\mathcal{G}_{3,3}$?

- Compute the ideal of the universal variety $\mathcal{U}_{3,3}$ in $\mathbb{P}^{26}$. What is its dimension, degree, singularities, Hilbert polynomial, ...?

- List explicit tensors in $\mathcal{U}_{3,3}$. Find corresponding paths in $\mathbb{R}^3$.