Varieties of Signature Tensors

Second Lecture

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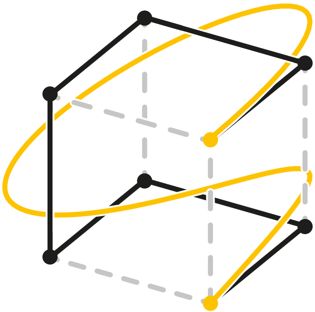

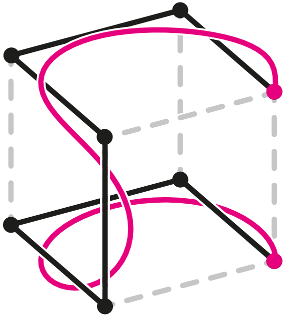

Polynomial Signature Varieties

Consider paths $X : [0,1] \to \mathbb{R}^d$ whose coordinates are polynomials of degree $\leq m$. We identify paths with $d \times m$-matrices $X = (x_{ij})$:

$$X_i(t) \;=\; x_{i1}t + x_{i2}t^2 + x_{i3}t^3 + \cdots + x_{im}t^m.$$

The kth signature $\sigma^{(k)}(X)$ is a $d \times d \times \cdots \times d$ tensor. It can be computed by multiplying our favorite $m \times m \times \cdots \times m$ tensor

$$\sigma^k(C_{\mathrm{mono}}) \;=\; \left[\frac{i_1}{i_1} \cdot \frac{i_2}{i_1+i_2} \cdot \frac{i_3}{i_1+i_2+i_3} \cdots \frac{i_k}{i_1+i_2+\cdots+i_k}\right]$$

on all k sides with the $d \times m$ matrix X.

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on all k sides with the $d \times m$ matrix X.

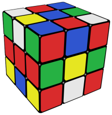
The *polynomial signature variety* $\mathcal{P}_{d,k,m}$ is the Zariski closure of the image of the rational map

$$\sigma^{(k)} \; : \; \mathbb{P}^{dm-1} \dashrightarrow \mathbb{P}^{d^k-1}, \; X \mapsto \sigma^{(k)}(X).$$

Remark: If $m \leq d$ then this is the closure of a $\mathrm{GL}(d)$ orbit in $(\mathbb{C}^d)^{\otimes m}$.

Example: Quadratic Paths in 3-Space

The third signature variety $\mathcal{P}_{3,3,2}$ for quadratic paths in $\mathbb{R}^3$ lies in the universal variety $\mathcal{U}_{3,3}$ for $3\times3\times3$ tensors.



$\mathcal{P}_{3,3,2}$ has dimension 5, degree 90, and is cut out by 162 quadrics in $\mathbb{P}^{25}$. Recall that $\mathcal{U}_{3,3}$ has dimension 13, degree 24, and 81 quadrics.

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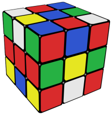

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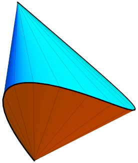
The linear span of $\mathcal{P}_{3,3,2}$ is the hyperplane $\mathbb{P}^{25}$ defined by

$$\sigma_{123} - \sigma_{132} - \sigma_{213} + \sigma_{231} + \sigma_{312} - \sigma_{321} \;=\; 0.$$



This linear form is the signed volume of the convex hull of a path.

Piecewise Linear Signature Varieties

Piecewise linear paths are also represented by $d \times m$ matrices X.

Their steps are the column vectors $X_1, \ldots, X_m \in \mathbb{R}^d$. The path is

$$t \mapsto X_1 + \cdots + X_{i-1} + (mt - i + 1) \cdot X_i \quad \text{for} \quad \tfrac{i-1}{m} \leq t \leq \tfrac{i}{m}.$$

The kth signature $\sigma^{(k)}(X)$ is a $d \times d \times \cdots \times d$ tensor. It can be computed by multiplying the upper triangular $m \times m \times \cdots \times m$ tensor $\sigma^k(C_{\text{axis}})$ on all k sides with the $d \times m$ matrix X.

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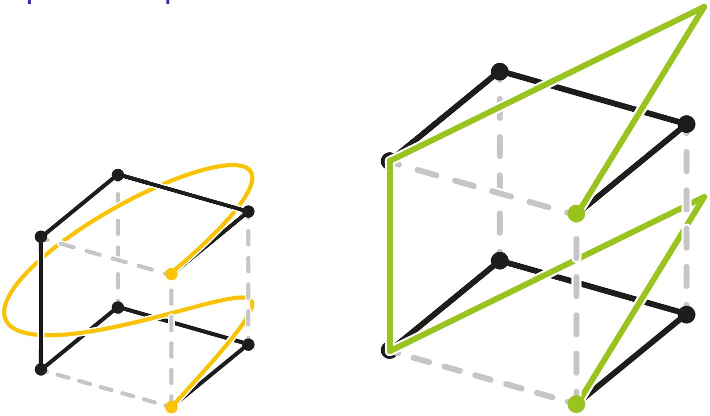
Few Steps in 3-Space



$$\sigma^{(3)}(X) \;=\; \frac{1}{6}\left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & -6 & 3 & 3 \\ 0 & 6 & 0 & -6 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

Parametrizations

By Chen (1954), the n-step signature of a piecewise linear path X is given by the tensor product of tensor exponentials:

$$\sigma^{\leq n}(X) \;=\; \exp(X_1) \otimes \exp(X_2) \otimes \cdots \otimes \exp(X_m) \;\in\; T^n(\mathbb{R}^d).$$

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Corollary

The kth signature tensor of X equals

$$\sigma^{(k)}(X) \;=\; \sum_{\tau} \prod_{\ell=1}^{m} \frac{1}{|\,\tau^{-1}(\ell)|\,!} \cdot X_{\tau(1)} \otimes X_{\tau(2)} \otimes X_{\tau(3)} \otimes \cdots \otimes X_{\tau(k)}.$$

Sum is over weakly increasing functions $\tau : \{1,\ldots,k\} \to \{1,\ldots,m\}$.

Example ($k = 3$)

The third signature is the $d\times d\times d$ tensor $\sigma^{(3)}(X) \;=$

$$\frac{1}{6}\cdot\sum_{i=1}^{m} X_i^{\otimes 3} \;+\; \frac{1}{2}\cdot\sum_{1\le i<j\le m} \left(X_i^{\otimes 2}\otimes X_j + X_i\otimes X_j^{\otimes 2}\right) \;+\; \sum_{1\le i<j<l\le m} X_i\otimes X_j\otimes X_k.$$

Inclusions

Theorem

For any d and any k, we have the following chains of inclusions between the kth Veronese variety and the kth universal variety:

$$\nu_k(\mathbb{P}^{d-1}) = \mathcal{L}_{d,k,1} \subset \mathcal{L}_{d,k,2} \subset \cdots \subset \mathcal{L}_{d,k,M-1} \subset \mathcal{L}_{d,k,M} = \mathcal{U}_{d,k} \subset \mathbb{P}^{d^k-1}$$
$$\nu_k(\mathbb{P}^{d-1}) = \mathcal{P}_{d,k,1} \subset \mathcal{P}_{d,k,2} \subset \cdots \subset \mathcal{P}_{d,k,M'-1} \subset \mathcal{P}_{d,k,M'} = \mathcal{U}_{d,k} \subset \mathbb{P}^{d^k-1}$$

Here M and M' are constants that depend only on d and k.

Remark

- *Dimension count yields conjectured values for M, M'.* More later.
- *The number m is similar to* **tensor rank**, *where a chain of secant varieties eventually fills the ambient space.*

Similarities and Differences

Polynomial and piecewise linear signature varieties agree for matrices:

$$\mathcal{L}_{d,2,m} \;=\; \mathcal{P}_{d,2,m}.$$

These are $d \times d$ matrices $P+Q$, where P is symmetric of rank ≤ 1, and Q is skew-symmetric, such that $\mathrm{rank}([\,P\,Q\,]) \leq m$.

Theorem

Two-segment paths and quadratic paths in $\mathbb{R}^2$ have different signature varieties $\mathcal{L}_{2,k,2} \neq \mathcal{P}_{2,k,2}$ in $\mathbb{P}^{2^k-1}$ for $k \geq 3$.

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Example ($k = 4$)

The threefolds $\mathcal{P}_{2,4,2}$ and $\mathcal{L}_{2,4,2}$ are orbit closures of $\mathrm{GL}(2)$ in $\mathbb{P}^{15}$.
We use invariant theory to distinguish these orbits.

The space of $\mathrm{SL}(2)$-invariant linear forms on $(\mathbb{R}^2)^{\otimes 4}$ is spanned by

$$\begin{array}{rrcl} & \ell_1 & = & \sigma_{1212} - \sigma_{1221} - \sigma_{2112} + \sigma_{2121} \\ \text{and} & \ell_2 & = & \sigma_{1122} - \sigma_{1221} - \sigma_{2112} + \sigma_{2211}. \end{array}$$

Their ratio ℓ_1/ℓ_2 is a rational function on $\mathbb{P}^{15}$ that is constant on orbits. It takes value 0 on C_{axis} and value $1/5$ on C_{mono}.

Data

We computed the polynomial and piecewise linear signature varieties for many tensor formats:

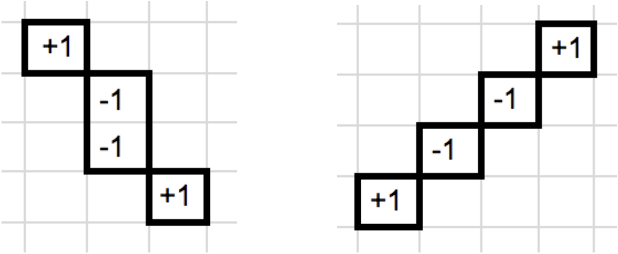
d	k	m	amb	dim	deg	gens
2	3	2	7	3	6	9
2	3	≥ 3	7	4	4	6
2	4	2	14	3	24	55
2	4	3	15	5	$192^{\mathcal{P}}, 64^{\mathcal{L}}$	$(33^{\mathcal{P}}, 34^{\mathcal{L}})$, $(0^{\mathcal{P}}, 3^{\mathcal{L}})$, ?
2	4	≥ 4	15	7	12	33
2	5	2	25	3	60	220
2	5	3	31	5	$1266^{\mathcal{P}}$, $492^{\mathcal{L}}$	$(160^{\mathcal{P}}, 185^{\mathcal{L}})$, ?
2	6	2	41	3	120	670
2	6	3	62	5	$4352^{\mathcal{P}}$, $1920^{\mathcal{L}}$	$(945^{\mathcal{P}}, 1056^{\mathcal{L}})$, ?
3	3	2	25	5	90	162
3	3	3	26	8	$756^{\mathcal{P}}, 396^{\mathcal{L}}$	$(83^{\mathcal{P}}, 91^{\mathcal{L}})$, ?
3	4	2	65	5	600	1536
3	4	3	80	8	?	$(1242^{\mathcal{P}}, 1374^{\mathcal{L}})$, ?

Table: Invariants of the varieties $\mathcal{P}_{d,k,m}$ and $\mathcal{L}_{d,k,m}$

A Question of Lyons and Xu

Proposition

There is an *axis path* *with* $m = 8$ *steps in alternating axis directions in the plane* $\mathbb{R}^2$ *and length* $l = 14 < 16 = 2^{k+1}$ *whose first* $k = 3$ *signature tensors are all zero.*



Answer to Question 2.5 in [T. Lyons and W. Xu: *Hyperbolic development and inversion of signature*, J. Funct. Anal. **272** (2017) 2933–2955]

Axis Paths

For *axis paths*, each step X_i is a multiple $a_i \cdot e_{\nu_i}$ of a basis vector. Record the step sequence $\nu = (\nu_1, \nu_2, \ldots, \nu_m) \in \{1, 2, \ldots, d\}^m$.

The kth signature tensors of such axis paths form a subvariety $\mathcal{A}_{\nu,k}$ of $\mathcal{L}_{d,k,m}$. It is parametrized by the lengths $a_1, a_2, \ldots, a_m$.

A current project by *Laura Colmenajero* and *Mateusz Michalek* studies the signature varieties $\mathcal{A}_{\nu,k}$. Stay tuned for their results.

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Example: For $d = 3$ and $\nu = (1, 2, 3)$ we get signature matrices

$$\sigma^{(2)}(X) \; = \; \frac{1}{2} \begin{pmatrix} a_1^2 & 2a_1a_2 & 2a_1a_3 \\ 0 & a_2^2 & 2a_2a_3 \\ 0 & 0 & a_3^2 \end{pmatrix}$$

Thus the signature variety $\mathcal{A}_{\nu,2}$ is a Veronese surface in $\mathbb{P}^5 \subset \mathbb{P}^8$.

Exercise: Compute $\mathcal{A}_{\nu,4} \subset \mathbb{P}^{15}$ for $d = 2, m \leq 7$, and $\nu = (1, 2, 1, 2, \ldots)$.

Identifiability

Counting parameters gives an upper bound on the dimension of our signature varieties: $\lambda_{d,k} = \#$ Lyndon words

$$\begin{aligned} & \dim(\mathcal{L}_{d,k,m}) \leq \min\{\lambda_{d,k} - 1, dm - 1\}, \\ \text{and} \quad & \dim(\mathcal{P}_{d,k,m}) \leq \min\{\lambda_{d,k} - 1, dm - 1\}. \end{aligned}$$

If the dimension equals $dm - 1$ then the variety is *algebraically identifiable*. This means that, for some $r \in \mathbb{N}$, the map from $d \times m$ matrices X to signature tensors $\sigma^{(k)}(X)$ is r-to-1. If $r = 1$ then the map is *birational*, and the variety is *rationally identifiable*.

Conjecture

- *Both inequalities are equalities provided $d, m \geq 2$ and $k \geq 3$.*
- *Stabilization constants for filling the universal variety are*

$$M = M' = \left\lceil \frac{\lambda_{d,k}}{d} \right\rceil .$$

Filling the Universal Variety

$d \setminus k$	3	4	5	6	7	8	9
2	3	**4**	7	12	21	36	64
3	5	**11**	27	66	170	440	1168
4	8	23	74	241	826	2866	10146
5	11	41	166	682	2914	12664	56064
6	16	68	327	1616	8281	43246	229866

Table: The value $M = M'$ at which the signature varieties stabilize.

Example ($d = 3, k = 4, M = 11$)

Consider $3 \times 3 \times 3 \times 3$ signature tensors for paths in $\mathbb{R}^3$. The universal variety $\mathcal{U}_{3,4}$ has dimension 31 and degree 672 in $\mathbb{P}^{80}$.

The signature varieties $\mathcal{P}_{3,4,10}$ and $\mathcal{L}_{3,4,10}$ have dimension 30. They are divisors in

$$\mathcal{P}_{3,4,11} \;=\; \mathcal{L}_{3,4,11} \;=\; \mathcal{U}_{3,4}.$$

At the Borderline

Identifiability is delicate for $\lambda_{d,k} = md$, when the signature variety exactly fills the universal variety. We expect algebraic identifiability.

Example ($d = 2, k = 4, M = 4$)

The 7-dim'l variety $\mathcal{P}_{2,4,4} = \mathcal{L}_{2,4,4} = \mathcal{U}_{2,4}$ has degree 12 in $\mathbb{P}^{15}$. Have two parametrizations from the $\mathbb{P}^7$ of 2×4 matrices. The map from quartic paths is 48-to-1. From four-segment paths it is 4-to-1.

Consider the four-segment path in $\mathbb{R}^2$ given by

$$X = \begin{bmatrix} 29 & 15 & 13 & 2 \\ 23 & 26 & 6 & 27 \end{bmatrix}$$

Three other paths have the same $2 \times 2 \times 2 \times 2$ signature tensor:

$$\begin{bmatrix} 36.74838 & -17.80169 & 37.75532 & 2.29799 \\ 27.39596 & -9.82926 & 40.23084 & 24.20246 \end{bmatrix},$$

$$\begin{bmatrix} 102.16286 & -131.13298 & 85.92484 & 2.04528 \\ 104.55786 & -136.84738 & 86.56467 & 27.72484 \end{bmatrix},$$

$$\begin{bmatrix} 38.53237 & 38.8057 & -79.20533 & 60.86735 \\ 28.69523 & 82.7734 & -147.7839 & 118.3152 \end{bmatrix}.$$

Rational Identifiability

We believe that low-complexity paths can be recovered from their signature tensors whenever this is permitted by the dimensions.

Conjecture

Let $k \geq 3$ and take m strictly less than the threshold M at which the universal variety is expected to be filled. Then both of the signature varieties $\mathcal{P}_{d,k,m}$ and $\mathcal{L}_{d,k,m}$ are rationally identifiable.

Current best results:

Theorem

- *Rational identifiability holds for $m \leq 7$*
- *Algebraic identifiability holds for $m \leq 30$.*
- *Identifiability holds for $\mathcal{L}_{d,k,m}$ provided $m \leq d$.*

This relies on reduction to 3-way tensors.

Reductions

Proposition

Fix integers d, k, m that satisfy $d \geq m \geq 1$ and $k \geq 3$.

(a) *If $\mathcal{L}_{m,3,m}$ is rationally (resp. algebraically) identifiable then $\mathcal{L}_{d,k,m}$ is as well.*

(b) *If $\mathcal{P}_{m,3,m}$ is rationally (resp. algebraically) identifiable then $\mathcal{P}_{d,k,m}$ is as well.*

Proof.

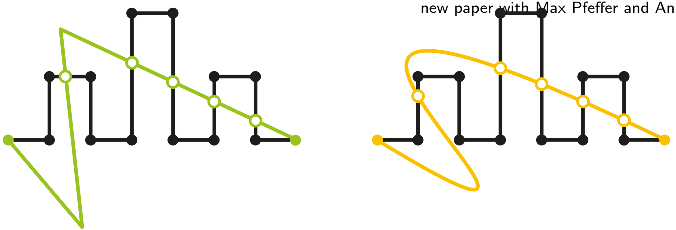
For the reduction from k to 3 we note that $\sigma^{(k)}(X)$ determines $\sigma^{(3)}(X)$ up to a multiplicative constant, by using shuffle relations.

The reduction from (d, m) to (m, m) is based on tensor methods. It relies on a variant of the Tucker decomposition (Kruskal's Theorem). The *core tensors* are $\sigma^k(C_{\mathrm{mono}})$ and $\sigma^k(C_{\mathrm{axis}})$. □

Invitation to read...

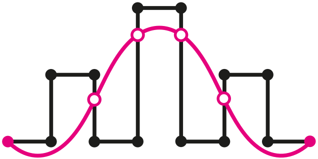
Learning Paths from Signature Tensors

new paper with Max Pfeffer and Anna Seigal



Abstract: *Matrix congruence extends naturally to the setting of tensors. We apply methods from tensor decomposition, algebraic geometry and numerical optimization to this group action. Given a tensor in the orbit of another tensor, we compute a matrix which transforms one to the other. Our primary application is an inverse problem from stochastic analysis: the recovery of paths from their signature tensors of order three. We establish identifiability results and recovery algorithms for piecewise linear paths, polynomial paths, and generic dictionaries. A detailed analysis of the relevant condition numbers is presented. We also compute the shortest path with a given signature tensor.*

Next summer in Bern



SIAM AG 19 Proposed Minisymposia

Algebraic methods in stochastic analysis

Organizers: Carlos Amendola and Anna Seigal

Signature tensors of paths

Organizers: Joscha Diehl and Francesco Galuppi